

# BANACH SPACES WITHOUT MINIMAL SUBSPACES - EXAMPLES

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ABSTRACT. We analyse several examples of separable Banach spaces, some of them new, and relate them to several dichotomies obtained in [11], by classifying them according to which side of the dichotomies they fall.

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## 1. INTRODUCTION

In this article we give several new examples of Banach spaces, corresponding to different classes of a list defined in [11]. This paper may be seen as a more empirical continuation of [11] in which our stress is on the study of examples for the new classes of Banach spaces considered in that work.

**1.1. Gowers' list of inevitable classes.** In the paper [15], W.T. Gowers had defined a program of isomorphic classification of Banach spaces. The aim of this program is a *loose classification of Banach spaces up to subspaces*, by producing a list of classes of Banach spaces such that:

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*Date:* May 2009.

*2000 Mathematics Subject Classification.* Primary: 46B03, Secondary 03E15.

*Key words and phrases.* Tight Banach spaces, Dichotomies, Classification of Banach spaces.

The first author acknowledges the support of FAPESP grant 2008/11471-6 and the second author the support of NSF grant DMS 0556368.

- (a) if a space belongs to a class, then every subspace belongs to the same class, or maybe, in the case when the properties defining the class depend on a basis of the space, every block subspace belongs to the same class,
- (b) the classes are *inevitable*, i.e., every Banach space contains a subspace in one of the classes,
- (c) any two classes in the list are disjoint,
- (d) belonging to one class gives a lot of information about operators that may be defined on the space or on its subspaces.

We shall refer to such a list as a *list of inevitable classes of Gowers*. For the motivation of Gowers' program as well as the relation of this program to classical problems in Banach space theory we refer to [11]. Let us just say that the class of spaces  $c_0$  and  $\ell_p$  is seen as the nicest or most regular class, and so, the objective of Gowers' program really is the classification of those spaces (such as Tsirelson's space  $T$ ) which do not contain a copy of  $c_0$  or  $\ell_p$ . Actually, in [11], mainly spaces without *minimal subspaces* are classified, and so in this article, we shall consider various examples of Banach spaces without minimal subspaces. We shall first give a summary of the classification obtained in [11] and of the results that led to that classification.

After the construction by Gowers and Maurey of a *hereditarily indecomposable* (or HI) space  $GM$ , i.e., a space such that no subspace may be written as the direct sum of infinite dimensional subspaces [16], Gowers proved that every Banach space contains either an HI subspace or a subspace with an unconditional basis [14]. This dichotomy is called *first dichotomy* of Gowers in [11]. These were the first two examples of inevitable classes. He then refined the list by proving a *second dichotomy*: any Banach space contains a subspace with a basis such that either no two disjointly supported block subspaces are isomorphic, or such that any two subspaces have further subspaces which are isomorphic. He called the second property *quasi minimality*. Finally, H. Rosenthal had defined a space to be *minimal* if it embeds into any of its subspaces. A quasi minimal space which does not contain a minimal subspace is called *strictly quasi minimal*, so Gowers again divided the class of quasi minimal spaces into the class of strictly quasi minimal spaces and the class of minimal spaces.

Gowers therefore produced a list of four inevitable classes of Banach spaces, corresponding to classical examples, or more recent counterexamples to classical questions: HI spaces, such as  $GM$ ; spaces with bases such that no disjointly supported subspaces are isomorphic, such as the counterexample  $G_u$  of Gowers to the hyperplane's problem of Banach [12]; strictly quasi minimal spaces with an unconditional basis, such as Tsirelson's space  $T$  [21]; and finally, minimal spaces, such as  $c_0$  or  $\ell_p$ , but also  $T^*$ , Schlumprecht's space  $S$  [19], or as proved recently in [18], its dual  $S^*$ .

**1.2. The three new dichotomies.** In [11] three dichotomies for Banach spaces were obtained. The first one of these new dichotomies, the *third dichotomy*, concerns the property of minimality defined by Rosenthal. Recall that a Banach space is minimal if it embeds into any of its infinite dimensional subspaces. On the other hand, a space  $Y$  is *tight* in a basic sequence  $(e_i)$  if there is a sequence of successive

subsets  $I_0 < I_1 < I_2 < \dots$  of  $\mathbb{N}$ , such that for all infinite subsets  $A \subseteq \mathbb{N}$ , we have

$$Y \not\subseteq [e_n \mid n \notin \bigcup_{i \in A} I_i].$$

A *tight basis* is a basis such that every subspace is tight in it, and a *tight space* is a space with a tight basis [11].

The subsets  $I_n$  may clearly be chosen to be intervals or even to form a partition of  $\mathbb{N}$ . However it is convenient not to require this condition in the definition, in view of forthcoming special cases of tightness.

It is observed in [11] that the tightness property is hereditary, incompatible with minimality, and it is proved that:

**Theorem 1.1** (3rd dichotomy, Ferenczi-Rosendal 2007). *Let  $E$  be a Banach space without minimal subspaces. Then  $E$  has a tight subspace.*

Actual examples of tight spaces in [11] turn out to satisfy one of two stronger forms of tightness. The first was called *tightness by range*. Here the range, range  $x$ , of a vector  $x$  is the smallest interval of integers containing its support on the given basis, and the range of a block subspace  $[x_n]$  is  $\bigcup_n \text{range } x_n$ . A basis  $(e_n)$  is tight by range when for every block subspace  $Y = [y_n]$ , the sequence of successive subsets  $I_0 < I_1 < \dots$  of  $\mathbb{N}$  witnessing the tightness of  $Y$  in  $(e_n)$  may be defined by  $I_k = \text{range } y_k$  for each  $k$ . This is equivalent to no two block subspaces with disjoint ranges being comparable, where two spaces are comparable if one embeds into the other.

When the definition of tightness may be checked with  $I_k = \text{supp } y_k$  instead of range  $y_k$ , then a stronger property is obtained which is called tightness by support, and is equivalent to the property defined by Gowers in the second dichotomy that no disjointly supported block subspaces are isomorphic, Therefore  $G_u$  is an example of space with a basis which is tight by support and therefore by range.

The second kind of tightness was called *tightness with constants*. A basis  $(e_n)$  is tight with constants when for every infinite dimensional space  $Y$ , the sequence of successive subsets  $I_0 < I_1 < \dots$  of  $\mathbb{N}$  witnessing the tightness of  $Y$  in  $(e_n)$  may be chosen so that  $Y \not\subseteq_K [e_n \mid n \notin I_K]$  for each  $K$ . This is the case for Tsirelson's space  $T$  or its  $p$ -convexified version  $T^{(p)}$  [8].

As we shall see, one of the aims of this paper is to present various examples of tight spaces of these two forms.

In [11] it was proved that there are natural dichotomies between each of these strong forms of tightness and respective weak forms of minimality. For the first notion, a space  $X$  with a basis  $(x_n)$  is said to be *subsequentially minimal* if every subspace of  $X$  contains an isomorphic copy of a subsequence of  $(x_n)$ . Essentially this notion had been previously considered by Kutzarova, Leung, Manoussakis and Tang in the context of modified partially mixed Tsirelson spaces [17].

**Theorem 1.2** (4th dichotomy, Ferenczi-Rosendal 2007). *Any Banach space  $E$  contains a subspace with a basis that is either tight by range or is subsequentially minimal.*

The second case in Theorem 1.2 may be improved to the following hereditary property of a basis  $(x_n)$ , that we call *sequential minimality*:  $(x_n)$  is quasi minimal and every block sequence of  $[x_n]$  has a subsequentially minimal block sequence.

There is also a dichotomy concerning tightness with constants. Recall that given two Banach spaces  $X$  and  $Y$ , we say that  $X$  is *crudely finitely representable* in  $Y$  if there is a constant  $K$  such that for any finite-dimensional subspace  $F \subseteq X$  there is an embedding  $T: F \rightarrow Y$  with constant  $K$ , i.e.,  $\|T\| \cdot \|T^{-1}\| \leq K$ . A space  $X$  is said to be *locally minimal* if for some constant  $K$ ,  $X$  is  $K$ -crudely finitely representable in any of its subspaces.

**Theorem 1.3** (5th dichotomy, Ferenczi-Rosendal 2007). *Any Banach space  $E$  contains a subspace with a basis that is either tight with constants or is locally minimal.*

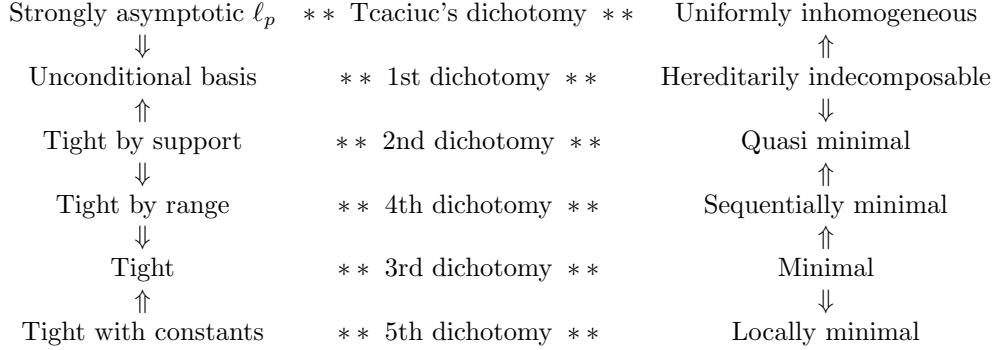
Finally there exists a sixth dichotomy theorem due to A. Tcaciuc [20], stated here in a slightly strengthened form. A space  $X$  is *uniformly inhomogeneous* when

$$\forall M \geq 1 \exists n \in \mathbb{N} \forall Y_1, \dots, Y_{2n} \subseteq X \exists y_i \in \mathcal{S}_{Y_i} (y_i)_{i=1}^n \not\sim_M (y_i)_{i=n+1}^{2n},$$

where  $Y_1, \dots, Y_{2n}$  are assumed to be infinite-dimensional subspaces of  $X$ . On the contrary, a basis  $(e_n)$  is said to be *strongly asymptotically  $\ell_p$* ,  $1 \leq p \leq +\infty$ , [9], if there exists a constant  $C$  and a function  $f: \mathbb{N} \rightarrow \mathbb{N}$  such that for any  $n$ , any family of  $n$  unit vectors which are disjointly supported in  $[e_k \mid k \geq f(n)]$  is  $C$ -equivalent to the canonical basis of  $\ell_p^n$ . Tcaciuc then proves [20] :

**Theorem 1.4** (Tcaciuc's dichotomy, 2005). *Any Banach space contains a subspace with a basis which is either uniformly inhomogeneous or strongly asymptotically  $\ell_p$  for some  $1 \leq p \leq +\infty$ .*

The six dichotomies and the interdependence of the properties involved can be visualised in the following diagram.



Moreover,

Strongly asymptotic  $\ell_p$  not containing  $\ell_p$ .  $1 \leq p < +\infty \Rightarrow$  Tight with constants,

and

Strongly asymptotic  $\ell_\infty \Rightarrow$  Locally minimal.

Note that while a basis tight by support must be unconditional, a basis which is tight by range may span a HI space. So tightness by support and tightness by range are two different notions. We would lose this subtle difference if we required the sets  $I_n$  to be intervals in the definition of tightness. Likewise a basis may be

tight by range without being (nor containing a basis which is) tight with constants, and tight with constants without being (nor containing a basis which is) tight by range. Actually none of the converses of the implications appearing on the left or the right of the list of the six dichotomies holds, even if one allows passing to a further subspace. All the claims of this paragraph are easily checked by looking at the list of examples of Theorem 1.5, which is the aim of this paper.

The fact that a strongly asymptotically  $\ell_p$  space not containing  $\ell_p$  must be tight with constants is proved in [11] but is essentially due to the authors of [9], and the observation that such bases are unconditional may also be found in [9]. The easy fact that HI spaces are uniformly homogeneous (with  $n = 2$  in the definition) is observed in [11]. That HI spaces are quasi-minimal is due to Gowers [15], and that minimal spaces are locally minimal is a consequence of an observation by P. G. Casazza [7] that every minimal space must  $K$ -embed into all its subspaces for some  $K \geq 1$ . The other implications are direct consequences of the definitions, and more explanations and details may be found in [11].

**1.3. The list of 19 inevitable classes.** Combining the six dichotomies and the relations between them, the following list of 19 classes of Banach spaces contained in any Banach space is obtained in [11]:

**Theorem 1.5** (Ferenczi - Rosendal 2007). *Any infinite dimensional Banach space contains a subspace of one of the types listed in the following chart:*

Type	Properties	Examples
(1a)	HI, tight by range and with constants	?
(1b)	HI, tight by range, locally minimal	$G^*$
(2)	HI, tight, sequentially minimal	?
(3a)	tight by support and with constants, uniformly inhomogeneous	?
(3b)	tight by support, locally minimal, uniformly inhomogeneous	$G_u^*$
(3c)	tight by support, strongly asymptotically $\ell_p$ , $1 \leq p < \infty$	$X_u, X_{abr}$
(3d)	tight by support, strongly asymptotically $\ell_\infty$	$X_u^*$
(4)	unconditional basis, quasi minimal, tight by range	?
(5a)	unconditional basis, tight with constants, sequentially minimal, uniformly inhomogeneous	?
(5b)	unconditional basis, tight, sequentially and locally minimal, uniformly inhomogeneous	?
(5c)	tight with constants, sequentially minimal, strongly asymptotically $\ell_p$ , $1 \leq p < \infty$	$T, T^{(p)}$
(5d)	tight, sequentially minimal, strongly asymptotically $\ell_\infty$	?
(6a)	unconditional basis, minimal, uniformly inhomogeneous	$S, S^*$
(6b)	minimal, reflexive, strongly asymptotically $\ell_\infty$	$T^*$
(6c)	isomorphic to $c_0$ or $\ell_p$ , $1 \leq p < \infty$	$c_0, \ell_p$

The class of type (2) spaces may be divided into two subclasses, using the 5th dichotomy, and the class of type (4) into four, using the 5th and the 6th dichotomy, giving a total of 19 inevitable classes. Since we know of no example of a type (2) or type (4) space to begin with, we do not write down the list of possible subclasses of these two classes, leaving this as an exercise to the interested reader.

Note that the tightness property may be used to obtain lower bounds of complexity for the relation of isomorphism between subspaces of a given Banach space. This was initiated by B. Bossard [6] who used Gowers' space  $G_u$  and its tightness by support. Other results in this direction may be found in [11]. We also refer to [10] for a more introductory work to this question.

In [11] the existence of  $X_u$  and the properties of  $S$ ,  $G$ ,  $G_u$  and  $X_u$  which appear in the chart and are mentioned without proof. It is the main objective of this paper to prove the results about the spaces which appear in the above chart.

So in what follows various (and for some of them new) examples of “pure” tight spaces are analysed combining some of the properties of tightness or minimality associated to each dichotomy. We shall provide several examples of tight spaces from the two main families of exotic Banach spaces: spaces of the type of Gowers and Maurey [16] and spaces of the type of Argyros and Deliyanni [3]. Recall that both types of spaces are defined using a coding procedure to “conditionalise” the norm of some ground space defined by induction. In spaces of the type of Gowers and Maurey, the ground space is the space  $S$  of Schlumprecht, and in spaces of the type of Argyros and Deliyanni, it is a mixed (in further versions modified or partly modified) Tsirelson space associated to the sequence of Schreier families. The space  $S$  is far from being asymptotic  $\ell_p$  and is actually uniformly inhomogeneous, and this is the case for our examples of the type of Gowers-Maurey as well. On the other hand, we use a space in the second family, inspired by an example of Argyros, Deliyanni, Kutzarova and Manoussakis [4], to produce strongly asymptotically  $\ell_1$  and  $\ell_\infty$  examples with strong tightness properties.

## 2. TIGHT UNCONDITIONAL SPACES OF THE TYPE OF GOWERS AND MAUREY

In this section we prove that the dual of the type (3) space  $G_u$  constructed by Gowers in [12] is locally minimal of type (3), that Gowers' hereditarily indecomposable and asymptotically unconditional space  $G$  defined in [13] is of type (1), and that its dual  $G^*$  is locally minimal of type (1). These spaces are natural variations on Gowers and Maurey's space  $GM$ , and so familiarity with that construction will be assumed: we shall not redefine the now classical notation relative to  $GM$ , such as the sets of integers  $K$  and  $L$ , rapidly increasing sequences (or R.I.S.), the set  $\mathbf{Q}$  of functionals, special functionals, etc., instead we shall try to give details on the parts in which  $G_u$  or  $G$  differ from  $GM$ .

The idea of the proofs is similar to [12]. The HI property for Gowers-Maurey's spaces is obtained as follows. Some vector  $x$  is constructed such that  $\|x\|$  is large, but so that if  $x'$  is obtained from  $x$  by changing signs of the components of  $x$ , then  $x^*(x')$  is small for any norming functional  $x^*$ , and so  $\|x'\|$  is small. The upper bound for  $x^*(x')$  is obtained by a combination of unconditional estimates (not depending on the signs) and of conditional estimates (i.e., based on the fact that  $|\sum_{i=1}^n \epsilon_i|$  is much smaller than  $n$  if  $\epsilon_i = (-1)^i$  for all  $i$ ).

For our examples we shall need to prove that some operator  $T$  is unbounded. Thus we shall construct a vector  $x$  such that say  $Tx$  has large norm, and such that  $x^*(x)$  is small for any norming  $x^*$ . The upper bound for  $x^*(x)$  will be obtained by the same unconditional estimates as in the HI case, while conditional estimates will

be trivial due to disjointness of supports of the corresponding component vectors and functionals. The method will be similar for the dual spaces.

Recall that if  $X$  is a space with a bimonotone basis, an  $\ell_{1+}^n$ -average with constant  $1+\epsilon$  is a normalised vector of the form  $\sum_{i=1}^n x_i$ , where  $x_1 < \dots < x_n$  and  $\|x_i\| \leq \frac{1+\epsilon}{n}$  for all  $i$ . An  $\ell_{\infty+}^n$ -average with constant  $1+\epsilon$  is a normalised vector of the form  $\sum_{i=1}^n x_i$ , where  $x_1 < \dots < x_n$  and  $\|x_i\| \geq \frac{1}{1+\epsilon}$  for all  $i$ . An  $\ell_{1+}^n$ -vector (resp.  $\ell_{\infty+}^n$ -vector) is a non zero multiple of an  $\ell_{1+}^n$ -average (resp.  $\ell_{\infty+}^n$ -average). The function  $f$  is defined by  $f(n) = \log_2(n+1)$ . The space  $X$  is said to satisfy a lower  $f$ -estimate if for any  $x_1 < \dots < x_n$ ,

$$\frac{1}{f(n)} \sum_{i=1}^n \|x_i\| \leq \left\| \sum_{i=1}^n x_i \right\|.$$

**Lemma 2.1.** *Let  $X$  be a reflexive space with a bimonotone basis and satisfying a lower  $f$ -estimate. Let  $(y_k^*)$  be a normalised block sequence of  $X^*$ ,  $n \in \mathbb{N}$ ,  $\epsilon, \alpha > 0$ . Then there exists a constant  $N(n, \epsilon)$ , successive subsets  $F_i$  of  $[1, N(n, \epsilon)]$ ,  $1 \leq i \leq n$ , and  $\lambda > 0$  such that if  $x_i^* := \lambda \sum_{k \in F_i} y_k^*$  for all  $i$ , then  $x^* = \sum_{i=1}^n x_i^*$  is an  $\ell_{\infty+}^n$ -average with constant  $1+\epsilon$ . Furthermore, if for each  $i$ ,  $x_i$  is such that  $\|x_i\| \leq 1$ ,  $\text{range } x_i \subseteq \text{range } x_i^*$  and  $x_i^*(x_i) \geq \alpha \|x_i^*\|$ , then  $x = \sum_{i=1}^n x_i$  is an  $\ell_{1+}^n$ -vector with constant  $\frac{1+\epsilon}{\alpha}$  such that  $x^*(x) \geq \frac{\alpha}{1+\epsilon} \|x\|$ .*

*Proof.* Since  $X$  satisfies a lower  $f$ -estimate, it follows by duality that any sequence of successive functionals  $x_1^* < \dots < x_n^*$  in  $G_u^*$  satisfies the following upper estimate:

$$1 \leq \left\| \sum_{i=1}^n x_i^* \right\| \leq f(n) \max_{1 \leq i \leq n} \|x_i^*\|.$$

Let  $N = n^k$  where  $k$  is such that  $(1+\epsilon)^k > f(n^k)$ . Assume towards a contradiction that the result is false for  $N(n, \epsilon) = N$ , then

$$y^* = (y_1^* + \dots + y_{n^{k-1}}^*) + \dots + (y_{(n-1)n^{k-1}+1}^* + \dots + y_{n^k}^*)$$

is not an  $\ell_{\infty+}^n$ -vector with constant  $1+\epsilon$ , and therefore, for some  $i$ ,

$$\|y_{in^{k-1}+1}^* + \dots + y_{(i+1)n^{k-1}}^*\| \leq \frac{1}{1+\epsilon} \|y^*\|.$$

Applying the same reasoning to the above sum instead of  $y^*$ , we obtain, for some  $j$ ,

$$\|y_{jn^{k-2}+1}^* + \dots + y_{(j+1)n^{k-2}}^*\| \leq \frac{1}{(1+\epsilon)^2} \|y^*\|.$$

By induction we obtain that

$$1 \leq \frac{1}{(1+\epsilon)^k} \|y^*\| \leq \frac{1}{(1+\epsilon)^k} f(n^k),$$

a contradiction.

Let therefore  $x^*$  be such an  $\ell_{\infty+}^n$ -average with constant  $1+\epsilon$  of the form  $\sum_i x_i^*$ . Let for each  $i$ ,  $x_i$  be such that  $\|x_i\| \leq 1$ ,  $\text{range } x_i \subseteq \text{range } x_i^*$  and  $x_i^*(x_i) \geq \alpha \|x_i^*\|$ . Then

$$\left\| \sum_i x_i \right\| \geq x^* \left( \sum_i x_i \right) \geq \alpha \left( \sum_i \|x_i^*\| \right) \geq \frac{\alpha n}{1+\epsilon},$$

and in particular for each  $i$ ,

$$\|x_i\| \leq 1 \leq \frac{1+\epsilon}{\alpha n} \left\| \sum_i x_i \right\|,$$

so  $\sum_i x_i$  is a  $\ell_{1+}^n$ -vector with constant  $\frac{1+\epsilon}{\alpha}$ . We also obtain that

$$x^* \left( \sum_i x_i \right) \geq \frac{\alpha n}{1+\epsilon} \geq \frac{\alpha}{1+\epsilon} \left\| \sum_i x_i \right\|,$$

as required.  $\square$

The following lemma is fundamental and therefore worth stating explicitly. It appears for example as Lemma 4 in [13]. Recall that an  $(M, g)$ -form is a functional of the form  $g(M)^{-1}(x_1^* + \dots + x_M^*)$ , with  $x_1^* < \dots < x_M^*$  of norm at most 1.

**Lemma 2.2** (Lemma 4 in [13]). *Let  $f, g \in \mathcal{F}$  with  $g \geq \sqrt{f}$ , let  $X$  be a space with a bimonotone basis satisfying a lower  $f$ -estimate, let  $\epsilon > 0$  and  $\epsilon' = \min\{\epsilon, 1\}$ , let  $x_1, \dots, x_N$  be a R.I.S. in  $X$  for  $f$  with constant  $1 + \epsilon$  and let  $x = \sum_{i=1}^N x_i$ . Suppose that*

$$\|Ex\| \leq \sup \left\{ |x^*(Ex)| : M \geq 2, x^* \text{ is an } (M, g)\text{-form} \right\}$$

for every interval  $E$  such that  $\|Ex\| \geq 1/3$ . Then  $\|x\| \leq (1 + \epsilon + \epsilon')Ng(N)^{-1}$ .

**2.1. A locally minimal space tight by support.** Let  $G_u$  be the space defined in [12]. This space has a suppression unconditional basis, is tight by support and therefore reflexive, and its norm is given by the following implicit equation, for all  $x \in c_{00}$ :

$$\begin{aligned} \|x\| = & \|x\|_{c_0} \vee \sup \left\{ f(n)^{-1} \sum_{i=1}^n \|E_i x\| \mid 2 \leq n, E_1 < \dots < E_n \right\} \\ & \vee \sup \left\{ |x^*(x)| \mid k \in K, x^* \text{ special of length } k \right\} \end{aligned}$$

where  $E_1, \dots, E_n$  are successive subsets (not necessarily intervals) of  $\mathbb{N}$ .

**Proposition 2.3.** *The dual  $G_u^*$  of  $G_u$  is tight by support and locally minimal.*

*Proof.* Given  $n \in \mathbb{N}$  and  $\epsilon = 1/10$  we may by Lemma 2.1 find some  $N$  such that there exists in the span of any  $x_1^* < \dots < x_N^*$  an  $\ell_{\infty+}^n$ -average with constant  $1 + \epsilon$ . By unconditionality we deduce that any block-subspace of  $G_u^*$  contains  $\ell_{\infty}^n$ 's uniformly, and therefore  $G_u^*$  is locally minimal.

Assume now towards a contradiction that  $(x_n^*)$  and  $(y_n^*)$  are disjointly supported and equivalent block sequences in  $G_u^*$ , and let  $T : [x_n^*] \rightarrow [y_n^*]$  be defined by  $Tx_n^* = y_n^*$ .

We may assume that each  $x_n^*$  is an  $\ell_{\infty+}^n$ -average with constant  $1 + \epsilon$ . Using Hahn-Banach theorem, the 1-unconditionality of the basis, and Lemma 2.1, we may also find for each  $n$  an  $\ell_{1+}^n$ -average  $x_n$  with constant  $1 + \epsilon$  such that  $\text{supp } x_n \subseteq \text{supp } x_n^*$  and  $x_n^*(x_n) \geq 1/2$ . By construction, for each  $n$ ,  $Tx_n^*$  is disjointly supported from  $[x_k]$ , and up to modifying  $T$ , we may assume that  $Tx_n^*$  is in  $\mathbf{Q}$  and of norm at most 1 for each  $n$ .



If  $z_1, \dots, z_m$  is a R.I.S. of these  $\ell_{1+}^n$ -averages  $x_n$  with constant  $1 + \epsilon$ , with  $m \in [\log N, \exp N]$ ,  $N \in L$ , and  $z_1^*, \dots, z_m^*$  are the functionals associated to  $z_1, \dots, z_m$ , then by [12] Lemma 7, the  $(m, f)$ -form  $z^* = f(m)^{-1}(z_1^* + \dots + z_m^*)$  satisfies

$$z^*(z_1 + \dots + z_m) \geq \frac{m}{2f(m)} \geq \frac{1}{4}\|z_1 + \dots + z_m\|,$$

and furthermore  $Tz^*$  is also an  $(m, f)$ -form. Therefore we may build R.I.S. vectors  $z$  with constant  $1 + \epsilon$  of arbitrary length  $m$  in  $[\log N, \exp N]$ ,  $N \in L$ , so that  $z$  is  $4^{-1}$ -normed by an  $(m, f)$ -form  $z^*$  such that  $Tz^*$  is also an  $(m, f)$ -form. We may then consider a sequence  $z_1, \dots, z_k$  of length  $k \in K$  of such R.I.S. vectors of length  $m_i$ , and some corresponding  $(m_i, f)$ -forms  $z_1^*, \dots, z_k^*$  (i.e.  $z_i^*$   $4^{-1}$ -norms  $z_i$  and  $Tz_i^*$  is also an  $(m_i, f)$ -form for all  $i$ ), such that  $Tz_1^*, \dots, Tz_k^*$  is a special sequence. Then we let  $z = z_1 + \dots + z_k$  and  $z^* = f(k)^{-1/2}(z_1^* + \dots + z_k^*)$ . Since  $Tz^* = f(k)^{-1/2}(Tz_1^* + \dots + Tz_k^*)$  is a special functional it follows that

$$\|Tz^*\| \leq 1.$$

Our aim is now to show that  $\|z\| \leq 3kf(k)^{-1}$ . It will then follow that

$$\|z^*\| \geq z^*(z)/\|z\| \geq f(k)^{1/2}/12.$$

Since  $k$  was arbitrary in  $K$  this will imply that  $T^{-1}$  is unbounded and provide the desired contradiction.

The proof is almost exactly the same as in [12]. Let  $K_0 = K \setminus \{k\}$  and let  $g$  be the corresponding function given by [12] Lemma 6. To prove that  $\|z\| \leq 3kf(k)^{-1}$  it is enough by [12] Lemma 8 and Lemma 2.2 to prove that for any interval  $E$  such that  $\|Ez\| \geq 1/3$ ,  $Ez$  is normed by some  $(M, g)$ -form with  $M \geq 2$ .

By the discussion in the proof of the main theorem in [12], the only possible norming functionals apart from  $(M, g)$ -forms are special functionals of length  $k$ . So let  $w^* = f(k)^{-1/2}(w_1^* + \dots + w_k^*)$  be a special functional of length  $k$ , and  $E$  be an interval such that  $\|Ez\| \geq 1/3$ . We need to show that  $w^*$  does not norm  $Ez$ .

Let  $t$  be minimal such that  $w_t^* \neq Tz_t^*$ . If  $i \neq j$  or  $i = j > t$  then by definition of special sequences there exist  $M \neq N \in L$ ,  $\min(M, N) \geq j_{2k}$ , such that  $w_i^*$  is an  $(M, f)$ -form and  $z_j$  is an R.I.S. vector of size  $N$  and constant  $1 + \epsilon$ . By [12] Lemma 8,  $z_j$  is an  $\ell_{1+}^{N^{1/10}}$ -average with constant 2. If  $M < N$  then  $2M < \log \log \log N$  so, by [12] Corollary 3,  $|w_i^*(Ez_j)| \leq 6f(M)^{-1}$ . If  $M > N$  then  $\log \log \log M > 2N$  so, by [12] Lemma 4,  $|w_i^*(Ez_j)| \leq 2f(N)/N$ . In both cases it follows that  $|w_i^*(Ez_j)| \leq k^{-2}$ .

If  $i = j = t$  we have  $|w_i^*(Ez_j)| \leq 1$ . Finally if  $i = j < t$  then  $w_i^* = Tz_i^*$ . Since  $Tz_i^*$  is disjointly supported from  $[x_k]$  and therefore from  $z_j$ , it follows simply that  $w_i^*(Ez_j) = 0$  in that case.

Summing up we have obtained that

$$|w^*(Ez)| \leq f(k)^{-1/2}(k^2 \cdot k^{-2} + 1) = 2f(k)^{-1/2} < 1/3 \leq \|Ez\|.$$

Therefore  $w^*$  does not norm  $Ez$  and this finishes the proof.  $\square$

**2.2. Uniformly inhomogeneous examples.** It may be observed that  $G_u^*$  is uniformly inhomogeneous. We state this in a general form which implies the result for  $G_u$ , Schlumprecht's space  $S$  and its dual  $S^*$ . This is also true for Gowers-Maurey's space  $GM$  and its dual  $GM^*$ , as well as for  $G$  and  $G^*$ , where  $G$  is the HI asymptotically unconditional space of Gowers from [13], which we shall redefine and study later on. As HI spaces are always uniformly inhomogeneous however, we need to observe that a slightly stronger result is obtained by the proof of the next statement

to see that Proposition 2.4 is not trivial in the case of  $GM$ ,  $G$  or their duals - see the three paragraphs after Proposition 2.4.

**Proposition 2.4.** *Let  $f \in \mathcal{F}$  and let  $X$  be a space with a bimonotone basis satisfying a lower  $f$ -estimate. Let  $\epsilon_0 = 1/10$ , and assume that for every  $n \in [\log N, \exp N]$ ,  $N \in L$ ,  $x_1, \dots, x_n$  a R.I.S. in  $X$  with constant  $1 + \epsilon_0$  and  $x = \sum_{i=1}^N x_i$ ,*

$$\|Ex\| \leq \sup \left\{ |x^*(Ex)| : M \geq 2, x^* \text{ is an } (M, f)\text{-form} \right\}$$

*for every interval  $E$  such that  $\|Ex\| \geq 1/3$ . Then  $X$  and  $X^*$  are uniformly inhomogeneous.*

*Proof.* Given  $\epsilon > 0$ , let  $m \in L$  be such that  $f(m) \geq 24\epsilon^{-1}$ . Let  $Y_1, \dots, Y_{2m}$  be arbitrary block subspaces of  $X$ . By the classical method for spaces with a lower  $f$  estimate, we may find a R.I.S. sequence  $y_1 < \dots < y_m$  with constant  $1 + \epsilon_0$  with  $y_i \in Y_{2i-1}, \forall i$ . By Lemma 2.2,

$$\left\| \sum_{i=1}^m y_i \right\| \leq 2mf(m)^{-1}.$$

Let on the other hand  $n \in [m^{10}, \exp m]$  and  $E_1 < \dots < E_m$  be sets such that  $\bigcup_{j=1}^m E_j = \{1, \dots, n\}$  and  $|E_j|$  is within 1 of  $\frac{n}{m}$  for all  $j$ . We may construct a R.I.S. sequence  $x_1, \dots, x_n$  with constant  $1 + \epsilon_0$  such that  $x_i \in Y_{2j}$  whenever  $i \in E_j$ .

By Lemma 2.2,

$$\left\| \sum_{i \in E_j} x_i \right\| \leq (1 + 2\epsilon_0) \left( \frac{n}{m} + 1 \right) f \left( \frac{n}{m} - 1 \right)^{-1} \leq 2nf(n)^{-1}m^{-1}.$$

Let  $z_j = \left\| \sum_{i \in E_j} x_i \right\|^{-1} \sum_{i \in E_j} x_i$ . Then  $z_j \in Y_{2j}$  for all  $j$  and

$$\left\| \sum_{j=1}^m z_j \right\| \geq f(n)^{-1} \sum_{j=1}^m \left( \left\| \sum_{i \in E_j} x_i \right\|^{-1} \sum_{i \in E_j} \|x_i\| \right) \geq m/2.$$

Therefore

$$\left\| \sum_{i=1}^m y_i \right\| \leq 4f(m)^{-1} \left\| \sum_{i=1}^m z_i \right\| \leq \epsilon \left\| \sum_{i=1}^m z_i \right\|.$$

Obviously  $(y_i)_{i=1}^m$  is not  $\epsilon^{-1}$ -equivalent to  $(z_i)_{i=1}^m$ , and this means that  $X$  is uniformly inhomogeneous.

The proof concerning the dual is quite similar and uses the same notation. Let  $Y_1, \dots, Y_{2m}$  be arbitrary block subspaces of  $X^*$ . By Lemma 2.1 we may find a R.I.S. sequence  $y_1 < \dots < y_m$  with constant  $1 + \epsilon_0$  and functionals  $y_i^* \in Y_{2i-1}$  such that  $\text{range } y_i^* \subseteq \text{range } y_i$  and  $y_i^*(y_i) \geq 1/2$  for all  $i$ . Since  $\left\| \sum_{i=1}^m y_i \right\| \leq 2mf(m)^{-1}$ , it follows that

$$\left\| \sum_{i=1}^m y_i^* \right\| \geq \left\| \sum_{i=1}^m y_i \right\|^{-1} \sum_{i=1}^m y_i^*(y_i) \geq f(m)/4.$$

On the other hand we may construct a R.I.S. sequence  $x_1, \dots, x_n$  with constant  $1 + \epsilon_0$  and functionals  $x_i^*$  such that  $\text{range } x_i^* \subseteq \text{range } x_i$ ,  $x_i^*(x_i) \geq 1/2$  for all  $i$ , and such that  $x_i^* \in Y_{2j}$  whenever  $i \in E_j$ . Since  $\left\| \sum_{i \in E_j} x_i \right\| \leq 2nf(n)^{-1}m^{-1}$ , it follows that

$$\left\| \sum_{i \in E_j} x_i^* \right\| \geq \frac{n}{3m} \frac{mf(n)}{2n} = f(n)/6.$$

Let  $z_j^* = \|\sum_{i \in E_j} x_i^*\|^{-1} \sum_{i \in E_j} x_i^*$ . Then  $z_j^* \in Y_{2j}$  for all  $j$  and

$$\left\| \sum_{j=1}^m z_j^* \right\| \leq \frac{6}{f(n)} f(n) = 6.$$

Therefore

$$\left\| \sum_{i=1}^m z_i^* \right\| \leq 24f(m)^{-1} \left\| \sum_{i=1}^m y_i^* \right\| \leq \epsilon \left\| \sum_{i=1}^m y_i^* \right\|.$$

□

**Corollary 2.5.** *The spaces  $S$ ,  $S^*$ ,  $GM$ ,  $GM^*$ ,  $G$ ,  $G^*$ ,  $G_u$ , and  $G_u^*$  are uniformly inhomogeneous.*

A slightly stronger statement may be obtained by the proof of Proposition 2.4, in the sense that the vectors  $y_i$  in the definition of uniform inhomogeneity may be chosen to be successive. More explicitly, the conclusion may be replaced by the statement that

$$\forall M \geq 1 \exists n \in \mathbb{N} \forall Y_1, \dots, Y_{2n} \subseteq X \exists y_i \in \mathcal{S}_{Y_i} (y_i)_{i=1}^n \not\sim_M (y_i)_{i=n+1}^{2n}.$$

where  $y_1 < \dots < y_n$  and  $y_{n+1} < \dots < y_{2n}$ , and as before  $Y_1, \dots, Y_{2n}$  are infinite-dimensional subspaces of  $X$ .

This property is therefore a block version of the property of uniform inhomogeneity. It was observed in [11] that the sixth dichotomy had the following “block” version: any Schauder basis of a Banach space contains a block sequence which is either block uniformly inhomogeneous in the above sense or asymptotically  $\ell_p$  for some  $p \in [1, +\infty]$ .

It is interesting to observe that either side of this dichotomy corresponds to one of the two main families of HI spaces, namely spaces of the type of Gowers-Maurey, based on the example of Schlumprecht, and spaces of the type of Argyros-Deliyanni, based on Tsirelson’s type spaces. More precisely, spaces of the type of Gowers-Maurey are block uniformly inhomogeneous, while spaces of the type of Argyros-Deliyanni are asymptotically  $\ell_1$ . Observe that the original dichotomy of Tcaciuc fails to distinguish between these two families, since any HI space is trivially uniformly inhomogeneous, see [11].

### 3. TIGHT HI SPACES OF THE TYPE OF GOWERS AND MAUREY

In this section we show that Gowers’ space  $G$  constructed in [13] and its dual are of type (1). The proof is a refinement of the proof that  $G_u$  or  $G_u^*$  is of type (3), in which we observe that the hypothesis of unconditionality may be replaced by asymptotic unconditionality. The idea is to produce constituent parts of vectors or functionals in Gowers’ construction with sufficient control on their supports (and not just on their ranges, as would be enough to obtain the HI property for example).

**3.1. A HI space tight by range.** The space  $G$  has a norm defined by induction as in  $GM$ , with the addition of a new term which guarantees that its basis  $(e_n)$  is 2-asymptotically unconditional, that is for any sequence of normalised vectors

$N < x_1 < \dots < x_N$ , any sequence of scalars  $a_1, \dots, a_N$  and any sequence of signs  $\epsilon_1, \dots, \epsilon_N$ ,

$$\left\| \sum_{n=1}^N \epsilon_n a_n x_n \right\| \leq 2 \left\| \sum_{n=1}^N a_n x_n \right\|.$$

The basis is bimonotone and, although this is not stated in [13], it may be proved as for  $GM$  that  $G$  is reflexive. It follows that the dual basis of  $(e_n)$  is also 2-asymptotically unconditional. The norm on  $G$  is defined by the implicit equation, for all  $x \in c_{00}$ :

$$\begin{aligned} \|x\| = & \|x\|_{c_0} \vee \sup \left\{ f(n)^{-1} \sum_{i=1}^n \|E_i x\| \mid 2 \leq n, E_1 < \dots < E_n \right\} \\ & \vee \sup \left\{ |x^*(Ex)| \mid k \in K, x^* \text{ special of length } k, E \subseteq \mathbb{N} \right\} \\ & \vee \sup \left\{ \|Sx\| \mid S \text{ is an admissible operator} \right\}, \end{aligned}$$

where  $E, E_1, \dots, E_n$  are intervals of integers, and  $S$  is an *admissible operator* if  $Sx = \frac{1}{2} \sum_{n=1}^N \epsilon_n E_n x$  for some sequence of signs  $\epsilon_1, \dots, \epsilon_N$  and some sequence  $E_1, \dots, E_N$  of intervals which is *admissible*, i.e.  $N < E_1$  and  $1 + \max E_i = \min E_{i+1}$  for every  $i < N$ .

*R.I.S. pairs* and *special pairs* are considered in [13]; first we shall need a more general definition of these. Let  $x_1, \dots, x_m$  be a R.I.S. with constant  $C$ ,  $m \in [\log N, \exp N]$ ,  $N \in L$ , and let  $x_1^*, \dots, x_m^*$  be successive normalised functionals. Then we call *generalised R.I.S. pair with constant  $C$*  the pair  $(x, x^*)$  defined by  $x = \|\sum_{i=1}^m x_i\|^{-1} (\sum_{i=1}^m x_i)$  and  $x^* = f(m)^{-1} \sum_{i=1}^m x_i^*$ .

Let  $z_1, \dots, z_k$  be a sequence of successive normalised R.I.S. vectors with constant  $C$ , and let  $z_1^*, \dots, z_k^*$  be a special sequence such that  $(z_i, z_i^*)$  is a generalized R.I.S. pair for each  $i$ . Then we shall call *generalised special pair with constant  $C$*  the pair  $(z, z^*)$  defined by  $z = \sum_{i=1}^k z_i$  and  $z^* = f(k)^{-1/2} (\sum_{i=1}^k z_i^*)$ . The pair  $(\|z\|^{-1} z, z^*)$  will be called *normalised generalised special pair*.

**Lemma 3.1.** *Let  $(z, z^*)$  be a generalised special pair in  $G$ , of length  $k \in K$ , with constant 2 and such that  $\text{supp } z^* \cap \text{supp } z = \emptyset$ . Then*

$$\|z\| \leq \frac{5k}{f(k)}.$$

*Proof.* The proof follows classically the methods of [16] or [12]. Let  $K_0 = K \setminus \{k\}$  and let  $g$  be the corresponding function given by [13] Lemma 5. To prove that  $\|z\| \leq 5kf(k)^{-1}$  it is enough by Lemma 2.2 to prove that for any interval  $E$  such that  $\|Ez\| \geq 1/3$ ,  $Ez$  is normed by some  $(M, g)$ -form with  $M \geq 2$ .

By the discussion in [13] after the definition of the norm, the only possible norming functionals apart from  $(M, g)$ -forms are of the form  $Sw^*$  where  $w^*$  is a special functional of length  $k$ , and  $S$  is an “acceptable” operator according to the terminology of [13]. We shall not state the definition of an acceptable operator  $S$ , we shall just need to know that since such an operator is diagonal of norm at most 1, it preserves support and  $(M, g)$ -forms, [13] Lemma 6. So let  $w^* = f(k)^{-1/2}(w_1^* + \dots + w_k^*)$  be a special functional of length  $k$ ,  $S$  be an acceptable operator, and  $E$  be an interval such that  $\|Ez\| \geq 1/3$ . We need to show that  $Sw^*$  does not norm  $Ez$ .

Let  $t$  be minimal such that  $w_t^* \neq z_t^*$ . If  $i \neq j$  or  $i = j > t$  then by definition of special sequences there exist  $M \neq N \in L$ ,  $\min(M, N) \geq j_{2k}$ , such that  $w_i^*$  and therefore  $Sw_i^*$  is an  $(M, f)$ -form and  $z_j$  is an R.I.S. vector of size  $N$  and constant 2. By [13] Lemma 8,  $z_j$  is an  $\ell_{1+}^{N^{1/10}}$ -average with constant 4. If  $M < N$  then  $2M < \log \log \log N$  so, by [13] Lemma 2,  $|Sw_i^*(Ez_j)| \leq 12f(M)^{-1}$ . If  $M > N$  then  $\log \log \log M > 2N$  so, by [13] Lemma 3,  $|Sw_i^*(Ez_j)| \leq 3f(N)/N$ . In both cases it follows that  $|Sw_i^*(Ez_j)| \leq k^{-2}$ .

If  $i = j = t$  we simply have  $|Sw_i^*(Ez_j)| \leq 1$ . Finally if  $i = j < t$  then  $w_i^* = z_i^*$ . and since  $\text{supp } Sz_i^* \subseteq \text{supp } z_i^*$  and  $\text{supp } Ez_i \subseteq \text{supp } z_i$ , it follows that  $Sw_i^*(Ez_j) = 0$  in this case.

Summing up we have obtained that

$$|Sw^*(Ez)| \leq f(k)^{-1/2}(k^2 \cdot k^{-2} + 1) = 2f(k)^{-1/2} < 1/3 \leq \|Ez\|.$$

Therefore  $Sw^*$  does not norm  $Ez$  and this finishes the proof.  $\square$

The next lemma is expressed in a version which may seem technical but this will make the proof that  $G$  is of type (1) more pleasant to read. At first reading, the reader may simply assume that  $T = Id$  in its hypothesis.

**Lemma 3.2.** *Let  $n \in \mathbb{N}$  and let  $\epsilon > 0$ . Let  $(x_i)_i$  be a normalised block basis in  $G$  of length  $n^k$  and supported after  $2n^k$ , where  $k = \min\{i \mid f(n^i) < (1 + \epsilon)^i\}$ , and  $T : [x_i] \rightarrow G$  be an isomorphism such that  $(Tx_i)$  is also a normalised block basis. Then for any  $n \in \mathbb{N}$  and  $\epsilon > 0$ , there exist a finite interval  $F$  and a multiple  $x$  of  $\sum_{i \in F} x_i$  such that  $Tx$  is an  $\ell_{1+}^n$ -average with constant  $1 + \epsilon$ , and a normalised functional  $x^*$  such that  $x^*(x) > 1/2$  and  $\text{supp } x^* \subseteq \bigcup_{i \in F} \text{range } x_i$ .*

*Proof.* The proof from [13] that the block basis  $(Tx_i)$  contains an  $\ell_{1+}^n$ -average with constant  $1 + \epsilon$  is the same as for  $GM$ , and gives that such a vector exists of the form  $Tx = \lambda \sum_{i \in F} Tx_i$ , thanks to the condition on the length of  $(x_i)$ . We may therefore deduce that  $2|F| - 1 < \text{supp } x$ . Let  $y^*$  be a unit functional which norms  $x$  and such that  $\text{range } y^* \subseteq \text{range } x$ . Let  $x^* = Ey^*$  where  $E$  is the union of the  $|F|$  intervals  $\text{range } x_i, i \in F$ . Then  $x^*(x) = y^*(x) = 1$  and by unconditional asymptoticity of  $G^*$ ,  $\|x^*\| \leq \frac{3}{2}\|y^*\| < 2$ .  $\square$

The proof that  $G$  is HI requires defining “extra-special sequences” after having defined special sequences in the usual  $GM$  way. However, to prove that  $G$  is tight by range, we shall not need to enter that level of complexity and shall just use special sequences.

**Proposition 3.3.** *The space  $G$  is of type (1).*

*Proof.* Assume some normalised block-sequence  $(x_n)$  is such that  $[x_n]$  embeds into  $Y = [e_i, i \notin \bigcup_n \text{range } x_n]$  and look for a contradiction. Passing to a subsequence and by reflexivity we may assume that there is some isomorphism  $T : [x_n] \rightarrow Y$  satisfying the hypothesis of Lemma 3.2, that is,  $(Tx_n)$  is a normalised block basis in  $Y$ . Fixing  $\epsilon = 1/10$  we may construct by Lemma 3.2 some block-sequence of vectors in  $[x_n]$  which are  $1/2$ -normed by functionals in  $\mathbf{Q}$  of support included in  $\bigcup_n \text{range } x_n$ , and whose images by  $T$  form a sequence of increasing length  $\ell_{1+}^n$ -averages with constant  $1 + \epsilon$ . If  $Tz_1, \dots, Tz_m$  is a R.I.S. of these  $\ell_{1+}^n$ -averages with constant  $1 + \epsilon$ , with  $m \in [\log N, \exp N]$ ,  $N \in L$ , and  $z_1^*, \dots, z_m^*$  are the

functionals associated to  $z_1, \dots, z_m$ , then by [13] Lemma 7, the  $(m, f)$ -form  $z^* = f(m)^{-1}(z_1^* + \dots + z_m^*)$  satisfies

$$z^*(z_1 + \dots + z_m) \geq \frac{m}{2f(m)} \geq \frac{1}{4} \|Tz_1 + \dots + Tz_m\| \geq (4\|T^{-1}\|)^{-1} \|z_1 + \dots + z_m\|.$$

Therefore we may build R.I.S. vectors  $Tz$  with constant  $1 + \epsilon$  of arbitrary length  $m$  in  $[\log N, \exp N]$ ,  $N \in L$ , so that  $z$  is  $(4\|T^{-1}\|)^{-1}$ -normed by an  $(m, f)$ -form  $z^*$  of support included in  $\bigcup_n \text{range } x_n$ . For such  $(z, z^*)$ ,  $(Tz, z^*)$  is a generalised R.I.S. pair. We then consider a sequence  $Tz_1, \dots, Tz_k$  of length  $k \in K$  of such R.I.S. vectors, such that there exists some special sequence of corresponding functionals  $z_1^*, \dots, z_k^*$ , and finally the pair  $(z, z^*)$  where  $z = z_1 + \dots + z_k$  and  $z^* = f(k)^{-1/2}(z_1^* + \dots + z_k^*)$ : observe that the support of  $z^*$  is still included in  $\bigcup_n \text{range } x_n$ . Since  $(Tz, z^*)$  is a generalised special pair, it follows from Lemma 3.1 that

$$\|Tz\| \leq 5kf(k)^{-1}.$$

On the other hand,

$$\|z\| \geq z^*(z) \geq (4\|T^{-1}\|)^{-1} kf(k)^{-1/2}.$$

Since  $k$  was arbitrary in  $K$  this implies that  $T^{-1}$  is unbounded and provides the desired contradiction.  $\square$

**3.2. A HI space tight by range and locally minimal.** As we shall now prove, the dual  $G^*$  of  $G$  is of type (1) as well, but also locally minimal.

**Lemma 3.4.** *Let  $(x_i^*)$  be a normalised block basis in  $G^*$ . Then for any  $n \in \mathbb{N}$  and  $\epsilon > 0$ , there exists  $N(n, \epsilon)$ , a finite interval  $F \subseteq [1, N(n, \epsilon)]$ , a multiple  $x^*$  of  $\sum_{i \in F} x_i^*$  which is an  $\ell_{\infty+}^n$ -average with constant  $1 + \epsilon$  and an  $\ell_{1+}^n$ -average  $x$  with constant 2 such that  $x^*(x) > 1/2$  and  $\text{supp } x \subseteq \bigcup_{i \in F} \text{range } x_i^*$ .*

*Proof.* We may assume that  $\epsilon < 1/6$ . By Lemma 2.1 we may find for each  $i \leq n$  an interval  $F_i$ , with  $|F_i| \leq 2 \min F_i$ , and a vector  $y_i^*$  of the form  $\lambda \sum_{k \in F_i} x_k^*$ , such that  $y^* = \sum_{i=1}^n y_i^*$  is an  $\ell_{\infty+}^n$ -average with constant  $1 + \epsilon$ . Let, for each  $i$ ,  $x_i$  be normalised such that  $y_i^*(x_i) = \|y_i^*\|$  and  $\text{range } x_i \subseteq \text{range } y_i^*$ . Let  $y_i = E_i x_i$ , where  $E_i$  denotes the canonical projection on  $[e_m, m \in \bigcup_{k \in F_i} \text{range } x_k^*]$ . By the asymptotic unconditionality of  $(e_n)$ , we have that  $\|y_i\| \leq 3/2$ . Let  $y'_i = \|y_i\|^{-1} y_i$ , then

$$y_i^*(y'_i) = \|y_i\|^{-1} y_i^*(y_i) = \|y_i\|^{-1} y_i^*(x_i) \geq \frac{2}{3} \|y_i^*\|.$$

By Lemma 2.1, the vector  $x = \sum_i y'_i$  is an  $\ell_{1+}^n$ -vector with constant 2, such that  $x^*(x) > \|x\|/2$ , and clearly  $\text{supp } x \subseteq \bigcup_{i \in F} \text{range } x_i^*$ .  $\square$

**Proposition 3.5.** *The space  $G^*$  is locally minimal and tight by range.*

*Proof.* By Lemma 3.4 we may find in any finite block subspace of  $G^*$  of length  $N(n, \epsilon)$  and supported after  $N(n, \epsilon)$  an  $\ell_{\infty+}^n$ -average with constant  $1 + \epsilon$ . By asymptotic unconditionality we deduce that uniformly, any block-subspace of  $G^*$  contains  $\ell_{\infty}^n$ 's, and therefore  $G^*$  is locally minimal.

We prove that  $G^*$  is tight by range. Assume towards a contradiction that some normalised block-sequence  $(x_n^*)$  is such that  $[x_n^*]$  embeds into  $Y = [e_i^*, i \notin \bigcup_n \text{range } x_n^*]$  and look for a contradiction. If  $T$  is the associated isomorphism, we may by passing to a subsequence and perturbing  $T$  assume that  $Tx_n^*$  is successive.

Let  $\epsilon = 1/10$ . By Lemma 3.4, we find in  $[x_k^*]$  and for each  $n$ , an  $\ell_{\infty+}^n$ -average  $y_n^*$  with constant  $1 + \epsilon$  and an  $\ell_{1+}^n$ -average  $y_n$  with constant 2, such that  $y_n^*(y_n) > 1/2$  and  $\text{supp } y_n \subseteq \bigcup_k \text{range } x_k^*$ . By construction, for each  $n$ ,  $T y_n^*$  is disjointly supported from  $[x_k^*]$ , and up to modifying  $T$ , we may assume that  $T y_n^*$  is in  $\mathbf{Q}$  and of norm at most 1 for each  $n$ .

If  $z_1, \dots, z_m$  is a R.I.S. of these  $\ell_{1+}^n$ -averages  $y_n$  with constant 2, with  $m \in [\log N, \exp N]$ ,  $N \in L$ , and  $z_1^*, \dots, z_m^*$  are the  $\ell_{\infty+}^n$ -averages associated to  $z_1, \dots, z_m$ , then by [12] Lemma 7, the  $(m, f)$ -form  $z^* = f(m)^{-1}(z_1^* + \dots + z_m^*)$  satisfies

$$z^*(z_1 + \dots + z_m) \geq \frac{m}{2f(m)} \geq \frac{1}{6} \|z_1 + \dots + z_m\|,$$

and furthermore  $Tz^*$  is also an  $(m, f)$ -form. Therefore we may build R.I.S. vectors  $z$  with constant 2 of arbitrary length  $m$  in  $[\log N, \exp N]$ ,  $N \in L$ , so that  $z$  is  $6^{-1}$ -normed by an  $(m, f)$ -form  $z^*$  such that  $Tz^*$  is also an  $(m, f)$ -form. We may then consider a sequence  $z_1, \dots, z_k$  of length  $k \in K$  of such R.I.S. vectors of length  $m_i$ , and some corresponding functionals  $z_1^*, \dots, z_k^*$  (i.e.,  $z_i^*$   $6^{-1}$ -norms  $z_i$  and  $Tz_i^*$  is also an  $(m_i, f)$ -form for all  $i$ ), such that  $Tz_1^*, \dots, Tz_k^*$  is a special sequence. Then we let  $z = z_1 + \dots + z_k$  and  $z^* = f(k)^{-1/2}(z_1^* + \dots + z_k^*)$ , and observe that  $(z, Tz^*)$  is a generalised special pair. Since  $Tz^* = f(k)^{-1/2}(Tz_1^* + \dots + Tz_k^*)$  is a special functional it follows that

$$\|Tz^*\| \leq 1.$$

But it follows from Lemma 3.1 that  $\|z\| \leq 5kf(k)^{-1}$ . Therefore

$$\|z^*\| \geq z^*(z)/\|z\| \geq f(k)^{1/2}/30.$$

Since  $k$  was arbitrary in  $K$  this implies that  $T^{-1}$  is unbounded and provides the desired contradiction.  $\square$

It remains to check that  $G^*$  is HI. The proof is very similar to the one in [13] that  $G$  is HI, and we shall therefore not give all details. There are two main differences between the two proofs. In [13] some special vectors and functionals are constructed, the vectors are taken alternatively in arbitrary block subspaces  $Y$  and  $Z$  of  $G$ , and no condition is imposed on where to pick the functionals. In our case there is no condition on where to choose the vectors but we need to pick the functionals in arbitrary subspaces  $Y$  and  $Z$  of  $G^*$  instead. This is possible because of Lemma 3.4. We also need to correct what seems to be a slight imprecision in the proof of [13] about the value of some normalising factors, and therefore we also get worst constants for our estimates.

Let  $\epsilon = 1/10$ . Following Gowers we define an *R.I.S. pair* of size  $N$  to be a generalised R.I.S. pair  $(x, x^*)$  with constant  $1 + \epsilon$  of the form  $(\|x_1 + \dots + x_N\|^{-1}(x_1 + \dots + x_N), f(N)^{-1}(x_1^* + \dots + x_N^*))$ , where  $x_n^*(x_n) \geq 1/3$  and  $\text{range } x_n^* \subseteq \text{range } x_n$  for each  $n$ . A *special pair* is a normalised generalised special pair with constant  $1 + \epsilon$  of the form  $(x, x^*)$  where  $x = \|x_1 + \dots + x_k\|^{-1}(x_1 + \dots + x_k)$  and  $x^* = f(k)^{-1/2}(x_1^* + \dots + x_k^*)$  with  $\text{range } x_n^* \subseteq \text{range } x_n$  and for each  $n$ ,  $x_n^* \in \mathbf{Q}$ ,  $|x_n^*(x_n) - 1/2| < 10^{-\min \text{supp } x_n}$ . By [13] Lemma 8,  $z$  is a R.I.S. vector with constant 2 whenever  $(z, z^*)$  is a special pair. We shall also require that  $k \leq \min \text{supp } x_1$ , which will imply by [13] Lemma 9 that for  $m < k^{1/10}$ ,  $z$  is a  $\ell_{1+}^m$ -average with constant 8 (see the beginning of the proof of Proposition 3.6).

Going up a level of “specialness”, a *special R.I.S.-pair* is a generalised R.I.S.-pair with constant 8 of the form  $(\|x_1 + \dots + x_N\|^{-1}(x_1 + \dots + x_N), f(N)^{-1}(x_1^* + \dots + x_N^*))$ ,

where  $\text{range } x_n^* \subset \text{range } x_n$  for each  $n$ , and with the additional condition that  $(x_n, x_n^*)$  is a special pair of length at least  $\min \text{supp } x_n$ . Finally, an *extra-special pair* of size  $k$  is a normalised generalised special pair  $(x, x^*)$  with constant 8 of the form  $x = \|x_1 + \dots + x_k\|^{-1}(x_1 + \dots + x_k)$  and  $x^* = f(k)^{-1/2}(x_1^* + \dots + x_k^*)$  with  $\text{range } x_n^* \subseteq \text{range } x_n$ , such that, for each  $n$ ,  $(x_n, x_n^*)$  is a special R.I.S.-pair of length  $\sigma(x_1^*, \dots, x_{n-1}^*)$ .

Given  $Y, Z$  block subspaces of  $G^*$  we shall show how to find an extra-special pair  $(x, x^*)$  of size  $k$ , with  $x^*$  built out of vectors in  $Y$  or  $Z$ , such that the signs of these constituent parts of  $x^*$  can be changed according to belonging to  $Y$  or  $Z$  to produce a vector  $x'^*$  with  $\|x'^*\| \leq 12f(k)^{-1/2}\|x^*\|$ . This will then prove the result.

Consider then an extra-special pair  $(x, x^*)$ . Then  $x$  splits up as

$$\nu^{-1} \sum_{i=1}^k \nu_i^{-1} \sum_{j=1}^{N_i} \nu_{ij}^{-1} \sum_{r=1}^{k_{ij}} x_{ijr}$$

and  $x^*$  as

$$f(k)^{-1/2} \sum_{i=1}^k f(N_i)^{-1} \sum_{j=1}^{N_i} f(k_{ij})^{-1} \sum_{r=1}^{k_{ij}} x_{ijr}^*$$

where the numbers  $\nu$ ,  $\nu_i$  and  $\nu_{ij}$  are the norms of what appears to the right. These special sequences are chosen far enough “to the right” so that  $k_{ij} \leq \min \text{supp } x_{ij1}$ , and also so that  $(\max \text{supp } x_{ij-1})^2 k_{ij}^{-1} \leq 4^{-(i+j)}$ . We shall also write  $x_i$  for  $\nu_i^{-1} \sum_{j=1}^{N_i} \nu_{ij}^{-1} \sum_{r=1}^{k_{ij}} x_{ijr}$  and  $x_{ij}$  for  $\nu_{ij}^{-1} \sum_{r=1}^{k_{ij}} x_{ijr}$ .

We define a vector  $x'$  by

$$\sum_{i=1}^k \nu_i'^{-1} \sum_{j=1}^{N_i} \nu_{ij}'^{-1} \sum_{r=1}^{k_{ij}} (-1)^r x_{ijr},$$

where the numbers  $\nu_i'$  and  $\nu_{ij}'$  are the norms of what appears to the right. We shall write  $x'_i$  for  $\nu_i'^{-1} \sum_{j=1}^{N_i} \nu_{ij}'^{-1} \sum_{r=1}^{k_{ij}} (-1)^r x_{ijr}$  and  $x'_{ij}$  for  $\nu_{ij}'^{-1} \sum_{r=1}^{k_{ij}} (-1)^r x_{ijr}$ .

Finally we define a functional  $x'^*$  as

$$f(k)^{-1/2} \sum_{i=1}^k f(N_i)^{-1} \sum_{j=1}^{N_i} f(k_{ij})^{-1} \sum_{r=1}^{k_{ij}} (-1)^r x_{ijr}^*.$$

**Proposition 3.6.** *The space  $G^*$  is HI.*

*Proof.* Fix  $Y$  and  $Z$  block subspaces of  $G^*$ . By Lemma 3.4 we may construct an extra-special pair  $(x, x^*)$  so that  $x_{ijr}^*$  belongs to  $Y$  when  $r$  is odd and to  $Z$  when  $r$  is even.

We first discuss the normalisation of the vectors involved in the definition of  $x'$ . By the increasing condition on  $k_{ij}$  and  $x_{ijr}$  and by asymptotic unconditionality, we have that

$$\left\| \sum_{r=1}^{k_{ij}} (-1)^r x_{ijr} \right\| \leq 2 \left\| \sum_{r=1}^{k_{ij}} x_{ijr} \right\|,$$

which means that  $\nu_{ij}' \leq 2\nu_{ij}$ . Furthermore it also follows that the functional  $(1/2)f(k_{ij})^{-1/2} \sum_{r=1}^{k_{ij}} (-1)^r x_{ijr}^*$  is of norm at most 1, and therefore we have that  $\left\| \sum_{r=1}^{k_{ij}} (-1)^r x_{ijr} \right\| \geq (1/2)k_{ij}f(k_{ij})^{-1/2}$ . Lemma 9 from [13] therefore tells us that,



for every  $i, j$ ,  $x'_{ij}$  is an  $\ell_{1+}^{m_{ij}}$ -average with constant 8, if  $m_{ij} < k_{ij}^{1/10}$ . But the  $k_{ij}$  increase so fast that, for any  $i$ , this implies that the sequence  $x'_{i1}, \dots, x'_{iN_i}$  is a rapidly increasing sequence with constant 8. By [13] Lemma 7, it follows that

$$\left\| \sum_{j=1}^{N_i} x'_{ij} \right\| \leq 9N_i/f(N_i).$$

Therefore by the  $f$ -lower estimate in  $G$  we have that  $\nu'_i \leq 9\nu_i$ .

We shall now prove that  $\|x'\| \leq 12kf(k)^{-1}$ . This will imply that

$$\begin{aligned} \|x'_*\| &\geq \frac{x'_*(x')}{\|x'\|} \geq \frac{f(k)}{12k} [f(k)^{-1/2} \sum_{i=1}^k f(N_i)^{-1} \nu_i'^{-1} \sum_{j=1}^{N_i} f(k_{ij})^{-1} \nu_{ij}'^{-1} \sum_{r=1}^{k_{ij}} x_{ijr}^*(x_{ijr})] \\ &\geq f(k)^{1/2} (12k)^{-1} \cdot 18^{-1} \left[ \sum_{i=1}^k f(N_i)^{-1} \nu_i'^{-1} \sum_{j=1}^{N_i} f(k_{ij})^{-1} \nu_{ij}'^{-1} \sum_{r=1}^{k_{ij}} x_{ijr}^*(x_{ijr}) \right] \\ &= f(k)^{1/2} (216k)^{-1} \sum_{i=1}^k x_i^*(x_i) \geq 648^{-1} f(k)^{1/2}. \end{aligned}$$

By construction of  $x^*$  and  $x'^*$  this will imply that

$$\|y^* - z^*\| \geq 648^{-1} f(k)^{1/2} \|y^* + z^*\|$$

for some non zero  $y^* \in Y$  and  $z^* \in Z$ , and since  $k \in K$  was arbitrary, as well as  $Y$  and  $Z$ , this will prove that  $G^*$  is HI.

The proof that  $\|x'\| \leq 12kf(k)^{-1}$  is given in three steps:

Step 1. *The vector  $x'$  is a R.I.S. vector with constant 11.*

*Proof.* We already know the sequence  $x'_{i1}, \dots, x'_{iN_i}$  is a rapidly increasing sequence with constant 8. Then by [13] Lemma 8 we get that  $x'_i$  is also an  $\ell_{1+}^{M_i}$ -average with constant 11, if  $M_i < N_i^{1/10}$ . Finally, this implies that  $x'$  is an R.I.S.-vector with constant 11, as claimed.  $\square$

Step 2. *Let  $K_0 = K \setminus \{k\}$ , let  $g \in \mathcal{F}$  be the corresponding function given by [13] Lemma 5. For every interval  $E$  such that  $\|Ex'\| \geq 1/3$ ,  $Ex'$  is normed by an  $(M, g)$ -form.*

*Proof.* The proof is exactly the same as the one of Step 2 in the proof of Gowers concerning  $G$ , apart from some constants which are modified due to the change of constant in Step 1 and to the normalising constants relating  $\nu_i$  and  $\nu_{ij}$  respectively to  $\nu'_i$  and  $\nu'_{ij}$ . The reader is therefore referred to [13].  $\square$

Step 3. *The norm of  $x'$  is at most  $12kg(k)^{-1} = 12kf(k)^{-1}$*

*Proof.* This is an immediate consequence of Step 1, Step 2 and of Lemma 2.2.  $\square$

We conclude that the space  $G^*$  is HI, and thus locally minimal of type (1).  $\square$

## 4. UNCONDITIONAL TIGHT SPACES OF THE TYPE OF ARGYROS AND DELIYANNI

By Proposition 2.4, unconditional or HI spaces built on the model of Gowers-Maurey's spaces are uniformly inhomogeneous (and even block uniformly inhomogeneous). We shall now consider a space of Argyros-Deliyanni type, more specifically of the type of a space constructed by Argyros, Deliyanni, Kutzarova and Manoussakis [4], with the opposite property, i.e., with a basis which is strongly asymptotically  $\ell_1$ . This space will also be tight by support and therefore will not contain a copy of  $\ell_1$ . By the implication at the end of the diagram which appears just before Theorem 1.5, this basis will therefore be tight with constants as well, making this example the “worst” known so far in terms of minimality.

Again in this section block vectors will not necessarily be normalized and some familiarity with the construction in [4] will be assumed.

**4.1. A strongly asymptotically  $\ell_1$  space tight by support.** In [4] an example of HI space  $X_{hi}$  is constructed, based on a “boundedly modified” mixed Tsirelson space  $X_{M(1),u}$ . We shall construct an unconditional version  $X_u$  of  $X_{hi}$  in a similar way as  $G_u$  is an unconditional version of  $GM$ . The proof that  $X_u$  is of type (3) will be based on the proof that  $X_{hi}$  is HI, conditional estimates in the proof of [4] becoming essentially trivial in our case due to disjointness of supports.

Fix a basis  $(e_n)$  and  $\mathcal{M}$  a family of finite subsets of  $\mathbb{N}$ . Recall that a family  $x_1, \dots, x_n$  is  $\mathcal{M}$ -admissible if  $x_1 < \dots < x_n$  and  $\{\min \text{supp } x_1, \dots, \min \text{supp } x_n\} \in \mathcal{M}$ , and  $\mathcal{M}$ -allowable if  $x_1, \dots, x_n$  are vectors with disjoint supports such that  $\{\min \text{supp } x_1, \dots, \min \text{supp } x_n\} \in \mathcal{M}$ . Let  $\mathcal{S}$  denote the family of Schreier sets, i.e., of subsets  $F$  of  $\mathbb{N}$  such that  $|F| \leq \min F$ ,  $\mathcal{M}_j$  be the subsequence of the sequence  $(\mathcal{F}_k)$  of Schreier families associated to sequences of integers  $t_j$  and  $k_j$  defined in [4] p 70.

We need to define a new notion. For  $W$  a set of functionals which is stable under projections onto subsets of  $\mathbb{N}$ , we let  $\text{conv}_{\mathbb{Q}} W$  denote the set of rational convex combinations of elements of  $W$ . By the stability property of  $W$  we may write any  $c^* \in \text{conv}_{\mathbb{Q}} W$  as a rational convex combination of the form  $\sum_i \lambda_i x_i^*$  where  $x_i^* \in W$  and  $\text{supp } x_i^* \subseteq \text{supp } c^*$  for each  $i$ . In this case the set  $\{x_i^*\}_i$  will be called a  $W$ -compatible decomposition of  $c^*$ , and we let  $W(c^*) \subseteq W$  be the union of all  $W$ -compatible decompositions of  $c^*$ . Note that if  $\mathcal{M}$  is a family of finite subsets of  $\mathbb{N}$ ,  $(c_1^*, \dots, c_d^*)$  is  $\mathcal{M}$ -admissible, and  $x_i^* \in W(c_i^*)$  for all  $i$ , then  $(x_1^*, \dots, x_d^*)$  is also  $\mathcal{M}$ -admissible.

Let  $\mathcal{B} = \{\sum_n \lambda_n e_n : (\lambda_n)_n \in c_{00}, \lambda_n \in \mathbb{Q} \cap [-1, 1]\}$  and let  $\Phi$  be a 1-1 function from  $\mathcal{B}^{<\mathbb{N}}$  into  $2\mathbb{N}$  such that if  $(c_1^*, \dots, c_k^*) \in \mathcal{B}^{<\mathbb{N}}$ ,  $j_1$  is minimal such that  $c_1^* \in \text{conv}_{\mathbb{Q}} \mathcal{A}_{j_1}$ , and  $j_l = \Phi(c_1^*, \dots, c_{l-1}^*)$  for each  $l = 2, 3, \dots$ , then  $\Phi(c_1^*, \dots, c_k^*) > \max\{j_1, \dots, j_k\}$  (the set  $\mathcal{A}_j$  is defined in [4] p 71 by  $\mathcal{A}_j = \cup_n (K_j^n \setminus K^0)$  where the  $K_j^n$ 's are the sets corresponding to the inductive definition of  $X_{M(1),u}$ ).

For  $j = 1, 2, \dots$ , we set  $L_j^0 = \{\pm e_n : n \in \mathbb{N}\}$ . Suppose that  $\{L_j^n\}_{j=1}^\infty$  have been defined. We set  $L^n = \cup_{j=1}^\infty L_j^n$  and

$$L_1^{n+1} = \pm L_1^n \cup \left\{ \frac{1}{2}(x_1^* + \dots + x_d^*) : d \in \mathbb{N}, x_i^* \in L^n, \right.$$

$$\left. (x_1^*, \dots, x_d^*) \text{ is } \mathcal{S} - \text{allowable} \right\},$$

and for  $j \geq 1$ ,

$$L_{2j}^{n+1} = \pm L_{2j}^n \cup \left\{ \frac{1}{m_{2j}}(x_1^* + \dots + x_d^*) : d \in \mathbb{N}, x_i^* \in L^n, \right. \\ \left. (x_1^*, \dots, x_d^*) \text{ is } \mathcal{M}_{2j} - \text{admissible} \right\},$$

$$L_{2j+1}'^{n+1} = \pm L_{2j+1}^n \cup \left\{ \frac{1}{m_{2j+1}}(x_1^* + \dots + x_d^*) : d \in \mathbb{N} \text{ such that} \right.$$

$\exists (c_1^*, \dots, c_d^*) \mathcal{M}_{2j+1} - \text{admissible and } k > 2j+1 \text{ with } c_1^* \in \text{conv}_{\mathbb{Q}} L_{2k}^n, x_1^* \in L_{2k}^n(c_1^*),$

$$c_i^* \in \text{conv}_{\mathbb{Q}} L_{\Phi(c_1^*, \dots, c_{i-1}^*)}^n, x_i^* \in L_{\Phi(c_1^*, \dots, c_{i-1}^*)}^n(c_i^*) \text{ for } 1 < i \leq d \},$$

$$L_{2j+1}^{n+1} = \{Ex^* : x^* \in L_{2j+1}'^{n+1}, E \text{ subset of } \mathbb{N}\}.$$

We set  $\mathcal{B}_j = \cup_{n=1}^{\infty} (L_j^n \setminus L^0)$  and we consider the norm on  $c_{00}$  defined by the set  $L = L^0 \cup (\cup_{j=1}^{\infty} \mathcal{B}_j)$ . The space  $X_u$  is the completion of  $c_{00}$  under this norm.

In [4] the space  $X_{hi}$  is defined in the same way except that  $E$  is an **interval** of integers in the definition of  $L_{2j+1}^{n+1}$ , and the definition of  $L_{2j+1}'^{n+1}$  is simpler, i.e., the coding  $\Phi$  is defined directly on  $\mathcal{M}_{2j+1}$ -admissible families  $x_1^*, \dots, x_d^*$  in  $L^{<\mathbb{N}}$  and in the definition each  $x_i^*$  belongs to  $L_{\Phi(x_1^*, \dots, x_{i-1}^*)}^n$ . To prove the desired properties for  $X_u$  one could use the simpler definition of  $L_{2j+1}'^{n+1}$ ; however this definition doesn't seem to provide enough special functionals to obtain interesting properties for the dual as well.

The ground space for  $X_{hi}$  and for  $X_u$  is the space  $X_{M(1),u}$  associated to a norming set  $K$  defined by the same procedure as  $L$ , except that  $K_{2j+1}^n$  is defined in the same way as  $K_{2j}^n$ , i.e.

$$K_{2j}^{n+1} = \pm K_{2j}^n \cup \left\{ \frac{1}{m_{2j}}(x_1^* + \dots + x_d^*) : d \in \mathbb{N}, x_i^* \in K^n, \right. \\ \left. (x_1^*, \dots, x_d^*) \text{ is } \mathcal{M}_{2j+1} - \text{admissible} \right\}.$$

For  $n = 0, 1, 2, \dots$ , we see that  $L_j^n$  is a subset of  $K_j^n$ , and therefore  $L \subseteq K$ . The norming set  $L$  is closed under projections onto **subsets** of  $\mathbb{N}$ , from which it follows that its canonical basis is unconditional, and has the property that for every  $j$  and every  $\mathcal{M}_{2j}$ -admissible family  $f_1, f_2, \dots, f_d$  contained in  $L$ ,  $f = \frac{1}{m_{2j}}(f_1 + \dots + f_d)$  belongs to  $L$ . The *weight* of such an  $f$  is defined by  $w(f) = 1/m_{2j}$ . It follows that for every  $j = 1, 2, \dots$  and every  $\mathcal{M}_{2j}$ -admissible family  $x_1 < x_2 < \dots < x_n$  in  $X_u$ ,

$$\left\| \sum_{k=1}^n x_k \right\| \geq \frac{1}{m_{2j}} \sum_{k=1}^n \|x_k\|.$$

Likewise, for  $\mathcal{S}$ -allowable families  $f_1, \dots, f_n$  in  $L$ , we have  $f = \frac{1}{2}(f_1 + \dots + f_d) \in L$ , and we define  $w(f) = 1/2$ . The weight is defined similarly in the case  $2j+1$ .

**Lemma 4.1.** *The canonical basis of  $X_u$  is strongly asymptotically  $\ell_1$ .*

*Proof.* Fix  $n \leq x_1, \dots, x_n$  where  $x_1, \dots, x_n$  are normalised and disjointly supported. Fix  $\epsilon > 0$  and let for each  $i$ ,  $f_i \in L$  be such that  $f_i(x_i) \geq (1 + \epsilon)^{-1}$  and  $\text{supp } f_i \subseteq \text{supp } x_i$ . The condition on the supports may be imposed because  $L$  is stable under projections onto subsets of  $\mathbb{N}$ . Then  $\frac{1}{2} \sum_{i=1}^n \pm f_i \in L$  and therefore

$$\left\| \sum_{i=1}^n \lambda_i x_i \right\| \geq \frac{1}{2} \sum_{i=1}^n |\lambda_i| f_i(x_i) \geq \frac{1}{2(1 + \epsilon)} \sum_{i=1}^n |\lambda_i|,$$

for any  $\lambda_i$ 's. Therefore  $x_1, \dots, x_n$  is 2-equivalent to the canonical basis of  $\ell_1^n$ .  $\square$

It remains to prove that  $X_u$  has type (3). Recall that an analysis  $(K^s(f))_s$  of  $f \in K$  is a decomposition of  $f$  corresponding to the inductive definition of  $K$ , see the precise definition in Definition 2.3 [4]. We shall combine three types of arguments. First  $L$  was constructed so that  $L \prec K$ , which means essentially that each  $f \in L$  has an analysis  $(K^s(f))_s$  whose elements actually belong to  $L$  (see the definition on page 74 of [4]); so all the results obtained in Section 2 of [4] for spaces defined through arbitrary  $\tilde{K} \prec K$  (and in particular the crucial Proposition 2.9) are valid in our case. Then we shall produce estimates similar to those valid for  $X_{hi}$  and which are of two forms: unconditional estimates, in which case the proofs from [4] may be applied directly up to minor changes of notation, and thus we shall refer to [4] for details of the proofs; and conditional estimates, which are different from those of  $X_{hi}$ , but easier due to hypotheses of disjointness of supports, and for which we shall give the proofs.

Recall that if  $\mathcal{F}$  is a family of finite subsets of  $\mathbb{N}$ , then

$$\mathcal{F}' = \{A \cup B : A, B \in \mathcal{F}, A \cap B = \emptyset\}.$$

Given  $\varepsilon > 0$  and  $j = 2, 3, \dots$ , an  $(\varepsilon, j)$ -basic special convex combination  $((\varepsilon, j)$ -basic s.c.c.) (relative to  $X_{M(1),u}$ ) is a vector of the form  $\sum_{k \in F} a_k e_k$  such that:  $F \in \mathcal{M}_j$ ,  $a_k \geq 0$ ,  $\sum_{k \in F} a_k = 1$ ,  $\{a_k\}_{k \in F}$  is decreasing, and, for every  $G \in \mathcal{F}'_{t_j(k_{j-1}+1)}$ ,  $\sum_{k \in G} a_k < \varepsilon$ .

Given a block sequence  $(x_k)_{k \in \mathbb{N}}$  in  $X_u$  and  $j \geq 2$ , a convex combination  $\sum_{i=1}^n a_i x_{k_i}$  is said to be an  $(\varepsilon, j)$ -special convex combination of  $(x_k)_{k \in \mathbb{N}}$   $((\varepsilon, j)$ -s.c.c), if there exist  $l_1 < l_2 < \dots < l_n$  such that  $2 < \text{supp } x_{k_1} \leq l_1 < \text{supp } x_{k_2} \leq l_2 < \dots < \text{supp } x_{k_n} \leq l_n$ , and  $\sum_{i=1}^n a_i e_{l_i}$  is an  $(\varepsilon, j)$ -basic s.c.c. An  $(\varepsilon, j)$ -s.c.c.  $\sum_{i=1}^n a_i x_{k_i}$  is called *seminormalised* if  $\|x_{k_i}\| = 1$ ,  $i = 1, \dots, n$  and

$$\left\| \sum_{i=1}^n a_i x_{k_i} \right\| \geq \frac{1}{2}.$$

Rapidly increasing sequences and  $(\varepsilon, j)$ -R.I. special convex combinations in  $X_u$  are defined by [4] Definitions 2.8 and 2.16 respectively, with  $\tilde{K} = L$ .

Using the lower estimate for  $\mathcal{M}_{2j}$ -admissible families in  $X_u$  we get as in [4] Lemma 3.1.

**Lemma 4.2.** *For  $\epsilon > 0$ ,  $j = 1, 2, \dots$  and every normalised block sequence  $\{x_k\}_{k=1}^\infty$  in  $X_u$ , there exists a finite normalised block sequence  $(y_s)_{s=1}^n$  of  $(x_k)$  and coefficients  $(a_s)_{s=1}^n$  such that  $\sum_{s=1}^n a_s y_s$  is a seminormalised  $(\epsilon, 2j)$ -s.c.c..*

The following definition is inspired from some of the hypotheses of [4] Proposition 3.3.

**Definition 4.3.** Let  $j > 100$ . Suppose that  $\{j_k\}_{k=1}^n$ ,  $\{y_k\}_{k=1}^n$ ,  $\{c_k^*\}_{k=1}^n$  and  $\{b_k\}_{k=1}^n$  are such that

(i) There exists a rapidly increasing sequence

$$\{x_{(k,i)} : k = 1, \dots, n, i = 1, \dots, n_k\}$$

with  $x_{(k,i)} < x_{(k,i+1)} < x_{(k+1,l)}$  for all  $k < n$ ,  $i < n_k$ ,  $l \leq n_{k+1}$ , such that:

- (a) Each  $x_{(k,i)}$  is a seminormalised  $(\frac{1}{m_{j(k,i)}^4}, j_{(k,i)})$ -s.c.c. where, for each  $k$ ,  $2j_k + 2 < j_{(k,i)}$ ,  $i = 1, \dots, n_k$ .  
 (b) Each  $y_k$  is a  $(\frac{1}{m_{2j_k}^4}, 2j_k)$ -R.I.s.c.c. of  $\{x_{(k,i)}\}_{i=1}^{n_k}$  of the form  $y_k = \sum_{i=1}^{n_k} b_{(k,i)} x_{(k,i)}$ .  
 (c) The sequence  $\{b_k\}_{k=1}^n$  is decreasing and  $\sum_{k=1}^n b_k y_k$  is a  $(\frac{1}{m_{2j+1}^4}, 2j+1)$ -s.c.c.  
 (ii)  $c_k^* \in \text{conv}_{\mathbb{Q}} L_{2j_k}$ , and  $\max(\text{supp } c_{k-1}^* \cup \text{supp } y_{k-1}) < \min(\text{supp } c_k^* \cup \text{supp } y_k)$ ,  $\forall k$ .

(iii)  $j_1 > 2j + 1$  and  $2j_k = \Phi(c_1^*, \dots, c_{k-1}^*)$ ,  $k = 2, \dots, n$ .

Then  $(j_k, y_k, c_k^*, b_k)_{k=1}^n$  is said to be a  $j$ -quadruple.

The following proposition is essential. It is the counterpart of Lemma 3.1 for the space  $X_u$ .

**Proposition 4.4.** Assume that  $(j_k, y_k, c_k^*, b_k)_{k=1}^n$  is a  $j$ -quadruple in  $X_u$  such that  $\text{supp } c_k^* \cap \text{supp } y_k = \emptyset$  for all  $k = 1, \dots, n$ . Then

$$\left\| \sum_{k=1}^n b_k m_{2j_k} y_k \right\| \leq \frac{75}{m_{2j+1}^2}.$$

*Proof.* Our aim is to show that for every  $\varphi \in \cup_{i=1}^{\infty} \mathcal{B}_i$ ,

$$\varphi\left(\sum_{k=1}^n b_k m_{2j_k} y_k\right) \leq \frac{75}{m_{2j+1}^2}.$$

The proof is given in several steps.

**1st Case:**  $w(\varphi) = \frac{1}{m_{2j+1}}$ . Then  $\varphi$  has the form  $\varphi = \frac{1}{m_{2j+1}}(Ey_1^* + \dots + Ey_{k_2}^* + Ey_{k_2+1}^* + \dots + Ey_d^*)$  where  $E$  is a subset of  $\mathbb{N}$  and where  $y_k^* \in L_{2j_k}(c_k^*) \forall k \leq k_2$  and  $y_k^* \in L_{2j_k}(d_k^*) \forall k \geq k_2 + 1$ , with  $d_{k_2+1}^* \neq c_{k_2+1}^*$  (this is similar to the form of such a functional in  $X_{hi}$  but with the integer  $k_1$  defined there equal to 1 in our case).

If  $k \leq k_2$  then  $c_s^*$  and therefore  $y_s^*$  is disjointly supported from  $y_k$ , so  $Ey_s^*(y_k) = 0$  for all  $s$ , and therefore  $\varphi(y_k) = 0$ . If  $k = k_2 + 1$  then we simply have  $|\varphi(y_k)| \leq \|y_k\| \leq 17m_{2j_k}^{-1}$ , [4] Corollary 2.17. Finally if  $k > k_2 + 1$  then since  $\Phi$  is 1-1, we have that  $j_{k_2+1} \neq j_k$  and for all  $s = k_2 + 1, \dots, d$ ,  $d_s^*$  and therefore  $y_s^*$  belong to  $\mathcal{B}_{2t_s}$  with  $t_s \neq j_k$ . It is then easy to check that we may reproduce the proof of [4] Lemma 3.5, applied to  $Ey_1^*, \dots, Ey_d^*$ , to obtain the unconditional estimate

$$|\varphi(m_{2j_k} y_k)| \leq \frac{1}{m_{2j+1}^2}.$$

In particular instead of [4] Proposition 3.2, which is a reformulation of [4] Corollary 2.17 for  $X_{hi}$ , we simply use [4] Corollary 2.17 with  $\tilde{K} = L$ .

Summing up these estimates we obtain the desired result for the 1st Case.

**2nd Case:**  $w(\varphi) \leq \frac{1}{m_{2j+2}}$ . Then we get an unconditional estimate for the evaluation of  $\varphi(\sum_{k=1}^n b_k m_{2j_k} y_k)$  directly, reproducing the short proof of [4] Lemma 3.7, using again [4] Corollary 2.17 instead of [4] Proposition 3.2. Therefore

$$|\varphi(\sum_{k=1}^n b_k m_{2j_k} y_k)| \leq \frac{35}{m_{2j+2}} \leq \frac{35}{m_{2j+1}^2}.$$

**3rd Case:**  $w(\varphi) > \frac{1}{m_{2j+1}}$ . We have  $y_k = \sum_{i=1}^{n_k} b_{(k,i)} x_{(k,i)}$  and the sequence  $\{x_{(k,i)}, k = 1, \dots, n, i = 1, \dots, n_k\}$  is a R.I.S. w.r.t.  $L$ . By [4] Proposition 2.9 there exist a functional  $\psi \in K'$  (see the definition in [4] p 71) and blocks of the basis  $u_{(k,i)}, k = 1, \dots, n, i = 1, \dots, n_k$  with  $\text{supp } u_{(k,i)} \subseteq \text{supp } x_{(k,i)}, \|u_k\|_{\ell_1} \leq 16$  and such that

$$\begin{aligned} |\varphi(\sum_{k=1}^n b_k m_{2j_k} (\sum_{i=1}^{n_k} b_{(k,i)} x_{(k,i)}))| &\leq m_{2j_1} b_1 b_{(1,1)} + \psi(\sum_{k=1}^n b_k m_{2j_k} (\sum_{i=1}^{n_k} b_{(k,i)} u_{(k,i)})) + \frac{1}{m_{2j+2}^2} \\ &\leq \psi(\sum_{k=1}^n b_k m_{2j_k} (\sum_{i=1}^{n_k} b_{(k,i)} u_{(k,i)})) + \frac{1}{m_{2j+2}}. \end{aligned}$$

Therefore it suffices to estimate

$$\psi(\sum_{k=1}^n b_k m_{2j_k} (\sum_{i=1}^{n_k} b_{(k,i)} u_{(k,i)})).$$

In [4]  $\psi$  is decomposed as  $\psi_1 + \psi_2$  and different estimates are applied to  $\psi_1$  and  $\psi_2$ . Our case is easier as we may simply assume that  $\psi_1 = 0$  and  $\psi_2 = \psi$ . We shall therefore refer to some arguments of [4] concerning some  $\psi_2$  keeping in mind that  $\psi_2 = \psi$ .

Let  $D_1^k, \dots, D_4^k$  be defined as in [4] Lemma 3.11 (a). Then as in [4],

$$\bigcup_{p=1}^4 D_p^k = \bigcup_{i=1}^{n_k} \text{supp } u_{(k,i)} \cap \text{supp } \psi.$$

The proof that

$$(1) \quad \psi|_{\bigcup_k D_2^k} (\sum_k b_k m_{2j_k} (\sum_i b_{(k,i)} u_{(k,i)})) \leq \frac{1}{m_{2j+2}},$$

$$(2) \quad \psi|_{\bigcup_k D_3^k} (\sum_k b_k m_{2j_k} (\sum_i b_{(k,i)} u_{(k,i)})) \leq \frac{16}{m_{2j+2}},$$

and

$$(3) \quad \psi|_{\bigcup_k D_1^k} (\sum_k b_k m_{2j_k} (\sum_i b_{(k,i)} u_{(k,i)})) \leq \frac{1}{m_{2j+2}}.$$

may be easily reproduced from [4] Lemma 3.11. The case of  $D_4^k$  is slightly different from [4] and therefore we give more details. We claim

*Claim:* Let  $D = \bigcup_k D_4^k$ . Then

$$(4) \quad \psi|_D (\sum_k b_k m_{2j_k} (\sum_i b_{(k,i)} u_{(k,i)})) \leq \frac{64}{m_{2j+2}},$$

Once the claim is proved it follows by adding the estimates that the 3rd Case is proved, and this concludes the proof of the Proposition.

*Proof of the claim:* Recall that  $D_4^k$  is defined by

$$D_4^k = \{m \in \bigcup_{i=1}^{n_k} A_{(k,i)} : \text{for all } f \in \bigcup_s K^s(\psi) \text{ with } m \in \text{supp } f, w(f) \geq \frac{1}{m_{2j_k}} \text{ and}$$

$$\text{there exists } f \in \bigcup_s K^s(\psi) \text{ with } m \in \text{supp } f, w(f) = \frac{1}{m_{2j_k}} \text{ and}$$

$$\text{for every } g \in \bigcup_s K^s(\psi) \text{ with } \text{supp } f \subset \text{supp } g \text{ strictly, } w(g) \geq \frac{1}{m_{2j+1}}\}.$$

For every  $k = 1, \dots, n$ ,  $i = 1, \dots, n_k$  and every  $m \in \text{supp } u_{(k,i)} \cap D_4^k$ , there exists a unique functional  $f^{(k,i,m)} \in \bigcup_s K^s(\psi)$  with  $m \in \text{supp } f$ ,  $w(f) = \frac{1}{m_{2j_k}}$  and such that, for all  $g \in \bigcup_s K^s(\psi)$  with  $\text{supp } f \subseteq \text{supp } g$  strictly,  $w(g) \geq \frac{1}{m_{2j+1}}$ . By definition, for  $k \neq p$  and  $i = 1, \dots, n_k$ ,  $m \in \text{supp } u_{(k,i)}$ , we have  $\text{supp } f^{(k,i,m)} \cap D_4^p = \emptyset$ . Also, if  $f^{(k,i,m)} \neq f^{(k,r,n)}$ , then  $\text{supp } f^{(k,i,m)} \cap \text{supp } f^{(k,r,n)} = \emptyset$ .

For each  $k = 1, \dots, n$ , let  $\{f^{k,t}\}_{t=1}^{r_k} \subseteq \bigcup_s K^s(\varphi)$  be a selection of mutually disjoint such functionals with  $D_4^k = \bigcup_{t=1}^{r_k} \text{supp } f^{k,t}$ . For each such functional  $f^{k,t}$ , we set

$$a_{f^{k,t}} = \sum_{i=1}^{n_k} b_{(k,i)} \sum_{m \in \text{supp } f^{k,t}} a_m.$$

Then,

$$(5) \quad f^{k,t}(b_k m_{2j_k} (\sum_i b_{(k,i)} u_{(k,i)})) \leq b_k a_{f^{k,t}}.$$

We define as in [4] a functional  $g \in K'$  with  $|g|_{2j}^* \leq 1$  (see definition [4] p 71), and blocks  $u_k$  of the basis so that  $\|u_k\|_{\ell_1} \leq 16$ ,  $\text{supp } u_k \subseteq \bigcup_i \text{supp } u_{(k,i)}$  and

$$\psi|_{D_4} (\sum_k b_k m_{2j_k} (\sum_i b_{(k,i)} u_{(k,i)})) \leq g(2 \sum_k b_k u_k),$$

hence by [4] Lemma 2.4(b) we shall have the result.

For  $f = \frac{1}{m_q} \sum_{p=1}^d f_p \in \bigcup_s K^s(\psi|_{D_4})$  we set

$$J = \{1 \leq p \leq d : f_p = f^{k,t} \text{ for some } k = 1, \dots, n, t = 1, \dots, r_k\},$$

$$T = \{1 \leq p \leq d : \text{there exists } f^{k,t} \text{ with } \text{supp } f^{k,t} \subseteq \text{supp } f_p \text{ strictly}\}.$$

For every  $f \in \bigcup_s K^s(\psi|_{D_4})$  we shall define by induction a functional  $g_f$ , by  $g_f = 0$  when  $J \cup T = \emptyset$ , while if  $J \cup T \neq \emptyset$  we shall construct  $g_f$  with the following properties. Let  $D_f = \bigcup_{p \in J \cup T} \text{supp } f_p$  and  $u_k = \sum a_{f^{k,t}} e_{f^{k,t}}$ , where  $e_{f^{k,t}} = e_{\min \text{supp } f^{k,t}}$ , then:

- (a)  $\text{supp } g_f \subseteq \text{supp } f$ .
- (b)  $g_f \in K'$  and  $w(g_f) \geq w(f)$ ,
- (c)  $f|_{D_f} (\sum_k b_k m_{2j_k} (\sum_i b_{(k,i)} u_{(k,i)})) \leq g_f(2 \sum_k b_k u_k)$ .

Let  $s > 0$  and suppose that  $g_f$  have been defined for all  $f \in \bigcup_{t=0}^{s-1} K^t(\psi|_{D_4})$  and let  $f = \frac{1}{m_q} (f_1 + \dots + f_d) \in K^s(\psi|_{D_4}) \setminus K^{s-1}(\psi|_{D_4})$  where the family  $(f_p)_{p=1}^d$  is  $\mathcal{M}_q$ -admissible if  $q > 1$ , or  $\mathcal{S}$ -allowable if  $q = 1$ . The proofs of case (i) ( $1/m_q = 1/m_{2j_k}$  for some  $k \leq n$ ) and case (ii) ( $1/m_q > 1/m_{2j+1}$ ) are identical with [4] p 106. Assume therefore that case (iii) holds, i.e.,  $1/m_q = 1/m_{2j+1}$ . For the same reasons as in [4] we have that  $T = \emptyset$ .

Summing up we assume that  $f \in K^s(\psi|_{D_4}) \setminus K^{s-1}(\psi|_{D_4})$  is of the form

$$f = \frac{1}{m_{2j+1}} \sum_{p=1}^d f_p = \frac{1}{m_{2j+1}} (Ey_1^* + \dots + Ey_{k_2}^* + Ey_{k_2+1}^* + \dots + Ey_d^*),$$

where  $(y_i^*)_i$  is associated to  $(c_1^*, \dots, c_{k_2}^*, d_{k_2+1}^*, \dots)$  with  $d_{k_2+1}^* \neq c_{k_2+1}^*$ , that  $T = \emptyset$  and  $J \neq \emptyset$ , and it only remains to define  $g_f$  satisfying (a)(b)(c).

Now by the proof of [4] Proposition 2.9,  $\psi = \psi_\varphi$  was defined through the analysis of  $\varphi$ , in particular by [4] Remark 2.19 (a),

$$\psi = \frac{1}{m_{2j+1}} \sum_{k \in I} \psi_{Ey_k^*}$$

for some subset  $I$  of  $\{1, \dots, d\}$ . Furthermore, for  $l \in I$ ,  $l \leq k_2$  and  $1 \leq k \leq d$ ,  $\text{supp } Ey_l^* \cap \text{supp } x_k = \emptyset$ , therefore there is no functional in a family of type I and II w.r.t.  $\overline{x_k}$  of support included in  $\text{supp } Ey_l^*$  (see [4] Definition 2.11 p 77). This implies that  $D_{Ey_l^*} = \emptyset$  ([4] Definition p 85), and therefore that  $\psi_{Ey_l^*} = 0$  ([4] bottom of p 85).

For  $l \in I$ ,  $l > k_2 + 1$ , then since  $\Phi$  is 1 - 1,  $w(Ey_l^*) = w(Ed_l^*) \neq 1/m_{2j_k} \forall k$ . Therefore  $w(\psi_{Ey_l^*}) \neq 1/m_{2j_k} \forall k$ , [4] Remark 2.19 (a). Then by the definition of  $D_4^k$ ,  $\text{supp } \psi_{Ey_l^*} \cap D_4^k = \emptyset$  for all  $k$ .

Finally this means that  $\psi|_{D_4} = \frac{1}{m_{2j+1}} \psi_{Ey_{k_2+1}^*|_{D_4}}$  and  $J = \{k_2 + 1\}$ ,  $D_f = \text{supp } f_{k_2+1}$ . Write then  $f_{k_2+1} = f^{k_0, t}$  and set  $g_f = \frac{1}{2} e_{f_{k_2+1}}^*$ , therefore (a)(b) are trivially verified. It only remains to check (c). But by (5),

$$\begin{aligned} f|_{D_f} \left( \sum_k b_k m_{2j_k} \left( \sum_i b_{(k,i)} u_{(k,i)} \right) \right) &\leq b_{k_0} a_{f_{k_2+1}} \\ &= b_{k_0} a_{f_{k_2+1}} e_{f_{k_2+1}}^* (e_{f_{k_2+1}}) = g_f (2b_{k_0} a_{f_{k_2+1}} e_{f_{k_2+1}}) \\ &= g_f \left( 2 \sum_t b_{k_0} a^{f_{k_0, t}} e_{f_{k_0, t}} \right) = g_f \left( 2 \sum_k b_k u_k \right). \end{aligned}$$

So (c) is proved. Therefore  $g_f$  is defined for each  $f$  by induction, and the Claim is verified. This concludes the proof of the Proposition.  $\square$

**Proposition 4.5.** *The space  $X_u$  is of type (3).*

*Proof.* Assume towards a contradiction that  $T$  is an isomorphism from some block-subspace  $[x_n]$  of  $X_u$  into the subspace  $[e_i, i \notin \bigcup_n \text{supp } x_n]$ . We may assume that  $\max(\text{supp } x_n, \text{supp } Tx_n) < \min(\text{supp } x_{n+1}, \text{supp } Tx_{n+1})$  and  $\min \text{supp } x_n < \min \text{supp } Tx_n$  for each  $n$ , and by Lemma 4.2, that each  $x_n$  is a  $(\frac{1}{m_{2n}^4}, 2n)$  R.I.s.c.c. ([4] Definition 2.16). We may write

$$x_n = \sum_{t=1}^{p_n} a_{n,t} x_{n,t}$$

where  $(x_{n,1}, \dots, x_{n,p_n})$  is  $\mathcal{M}_{2n}$ -admissible. Let for each  $n, t$ ,  $x_{n,t}^* \in L$  be such that  $\text{supp } x_{n,t}^* \subseteq \text{supp } Tx_{n,t}$  and such that

$$x_{n,t}^*(Tx_{n,t}) \geq \frac{1}{2} \|Tx_{n,t}\| \geq \frac{1}{4\|T^{-1}\|},$$



and let  $x_n^* = \frac{1}{m_{2n}}(x_{n,1}^* + \dots + x_{n,p_n}^*) \in L_{2n}$ . Note that  $\text{supp } x_n^* \cap \text{supp } x_n = \emptyset$  and that

$$x_n^*(Tx_n) \geq \frac{1}{m_{2n}} \sum_{t=1}^{p_n} \frac{a_{n,t}}{4\|T^{-1}\|} = (4\|T^{-1}\|m_{2n})^{-1}.$$

We may therefore for any  $j > 100$  construct a  $j$ -quadruple  $(j_k, y_k, c_k^*, b_k)_{k=1}^n$  satisfying the hypotheses of Proposition 4.4 and such that  $y_k \in [x_i]_i$  and  $c_k^*(Ty_k) \geq (4\|T^{-1}\|m_{2j_k})^{-1}$  for each  $k$  (note that we may assume that  $c_k^* \in L_{j_{2k}}$  for each  $k$ ). From Proposition 4.4 we deduce

$$\left\| \sum_{k=1}^n b_k m_{2j_k} y_k \right\| \leq \frac{75}{m_{2j+1}^2}.$$

On the other hand  $\psi = \frac{1}{m_{2j+1}} \sum_{k=1}^n c_k^*$  belongs to  $L$  therefore

$$\|T(\sum_{k=1}^n b_k m_{2j_k} y_k)\| \geq \psi(\sum_{k=1}^n b_k m_{2j_k} Ty_k) \geq \frac{1}{4\|T^{-1}\|m_{2j+1}}.$$

We deduce finally that

$$m_{2j+1} \leq 300\|T\|\|T^{-1}\|,$$

which contradicts the boundedness of  $T$ .  $\square$

**4.2. A strongly asymptotically  $\ell_\infty$  space tight by support.** Since the canonical basis of  $X_u$  is tight and unconditional, it follows that  $X_u$  is reflexive. In particular this implies that the dual basis of the canonical basis of  $X_u$  is a strongly asymptotically  $\ell_\infty$  basis of  $X_u^*$ . It remains to prove that this basis is tight with support.

It is easy to prove by duality that for any  $\mathcal{M}_{2j}$ -admissible sequence of functionals  $f_1, \dots, f_n$  in  $X_u^*$ , we have the upper estimate

$$\left\| \sum_i f_i \right\| \leq m_{2j} \sup_i \|f_i\|.$$

We use this observation to prove a lemma about the existence of s.c.c. normed by functionals belonging to an arbitrary subspace of  $X_u^*$ . The proof is standard except that estimates have to be taken in  $X_u^*$  instead of  $X_u$ .

**Lemma 4.6.** *For  $\epsilon > 0$ ,  $j = 1, 2, \dots$  and every normalised block sequence  $\{f_k\}_{k=1}^\infty$  in  $X_u^*$ , there exists a normalised functional  $f \in [f_k]$  and a seminormalised  $(\epsilon, 2j)$ -s.c.c.  $x$  in  $X_u$  such that  $\text{supp } f \subseteq \text{supp } x$  and  $f(x) \geq 1/2$ .*

*Proof.* For each  $k$  let  $y_k$  be normalised such that  $\text{supp } y_k = \text{supp } f_k$  and  $f_k(y_k) = 1$ . Recall that the integers  $k_n$  and  $t_n$  are defined by  $k_1 = 1$ ,  $2^{t_n} \geq m_{k_n}^2$  and  $k_n = t_n(k_{n-1} + 1) + 1$ , and that  $\mathcal{M}_j = \mathcal{F}_{k_j}$  for all  $j$ .

Applying Lemma 4.2 we find a successive sequence of  $(\epsilon, 2j)$ -s.c.c. of  $(y_k)$  of the form  $(\sum_{i \in I_k} a_i y_i)_k$  with  $\{f_i, i \in I_k\}$   $\mathcal{F}_{k_{2j-1}+1}$ -admissible. If  $\|\sum_{i \in I_k} f_i\| \leq 2$  for some  $k$ , we are done, for then

$$(\sum_{i \in I_k} f_i)(\sum_{i \in I_k} a_i y_i) \geq \frac{1}{2} \|\sum_{i \in I_k} f_i\|.$$

So assume  $\|\sum_{i \in I_k} f_i\| > 2$  for all  $k$ , apply the same procedure to the sequence  $f_k^1 = \|\sum_{i \in I_k} f_i\|^{-1} \sum_{i \in I_k} f_i$ , and obtain a successive sequence of  $(\epsilon, 2j)$ -s.c.c. of the sequence  $(y_k^1)_k$  associated to  $(f_k^1)_k$ , of the form  $(\sum_{i \in I_k^1} a_i^1 y_i^1)_k$ , with  $\{f_l : \text{supp } f_l \subseteq$

$\sum_{i \in I_k^1} f_i^1$  a  $\mathcal{F}_{k_{2j-1}+1}[\mathcal{F}_{k_{2j-1}+1}]$ -admissible, and therefore  $\mathcal{M}_{2j}$ -admissible set. Then we are done unless  $\|\sum_{i \in I_k^1} f_i^1\| > 2$  for all  $k$ , in which case we set

$$f_k^2 = \left\| \sum_{j \in I_k^1} f_j^1 \right\|^{-1} \sum_{j \in I_k^1} f_j^1$$

and observe by the upper estimate in  $X_u^*$  that

$$1 = \|f_k^2\| = \left\| \sum_{j \in I_k^1} \sum_{i \in I_j} \left\| \sum_{j \in I_k^1} f_j^1 \right\|^{-1} f_i \right\| \leq m_{2j}/4.$$

Repeating this procedure we claim that we are done in at most  $t_{2j}$  steps. Otherwise we obtain that the set

$$A = \{f_l : \text{supp } f_l \subseteq \sum_{i \in I_k^{t_{2j}-1}} f_i^{t_{2j}-1}\}$$

is  $\mathcal{M}_{2j}$ -admissible. Since  $f_k^{t_{2j}} = \sum_{f_l \in A} \alpha_l f_l$ , where the normalising factor  $\alpha_l$  is less than  $(1/2)^{t_{2j}}$  for each  $l$ , we deduce from the upper estimate that

$$1 = \|f_k^{t_{2j}}\| \leq 2^{-t_{2j}} m_{2j},$$

a contradiction by definition of the integers  $t_i$ 's.  $\square$

To prove the last proposition of this section we need to make two observations. First if  $(f_1, \dots, f_n) \in \text{conv}_{\mathbb{Q}} L$  is  $\mathcal{M}_{2j}$ -admissible, then  $\frac{1}{m_{2j}} \sum_{k=1}^n f_k \in \text{conv}_{\mathbb{Q}} L_{2j}$ . Indeed using the stability of  $L$  under projections onto subsets of  $\mathbb{N}$  we may easily find convex rational coefficients  $\lambda_i$  such that each  $f_k$  is of the form

$$f_k = \sum_i \lambda_i f_i^k, \quad f_i^k \in L, \quad \text{supp } f_i^k \subseteq \text{supp } f_k \quad \forall i.$$

Then  $\frac{1}{m_{2j}} \sum_{k=1}^n f_k = \sum_i \lambda_i (\frac{1}{m_{2j}} \sum_{k=1}^n f_i^k)$  and each  $\frac{1}{m_{2j}} \sum_{k=1}^n f_i^k$  belongs to  $L_{2j}$ .

Likewise if  $\psi = \frac{1}{m_{2j+1}}(c_1^* + \dots + c_d^*)$ ,  $k > 2j + 1$ ,  $c_1^* \in \text{conv}_{\mathbb{Q}} L_{2k}$  and  $c_l^* \in \text{conv}_{\mathbb{Q}} L_{\Phi(c_1^*, \dots, c_{l-1}^*)}$   $\forall l \geq 2$ , then  $\psi \in \text{conv}_{\mathbb{Q}} L$ . Indeed as above we may write

$$\psi = \sum_i \lambda_i \left( \frac{1}{m_{2j+1}} \sum_{l=1}^d f_i^l \right), \quad f_i^1 \in L_{2k}, f_i^l \in L_{\Phi(c_1^*, \dots, c_{l-1}^*)}(c_i^*) \quad \forall l \geq 2,$$

and each  $\frac{1}{m_{2j+1}} \sum_{l=1}^d f_i^l$  belongs to  $L_{2j+1}^{n+1} \subseteq L$ .

**Proposition 4.7.** *The space  $X_u^*$  is of type (3).*

*Proof.* Assume towards a contradiction that  $T$  is an isomorphism from some block-subspace  $[f_n]$  of  $X_u^*$  into the subspace  $[e_i^*, i \notin \cup_n \text{supp } f_n]$ . We may assume that  $\max(\text{supp } f_n, \text{supp } T f_n) < \min(\text{supp } f_{n+1}, \text{supp } T f_{n+1})$  and  $\min \text{supp } T f_n < \min \text{supp } f_n$  for each  $n$ . Since the closed unit ball of  $X_u^*$  is equal to  $\overline{\text{conv}_{\mathbb{Q}} L}$  we may also assume that  $f_n \in \text{conv}_{\mathbb{Q}} L$  for each  $n$ . Applying Lemma 4.6, we may also suppose that each  $f_n$  is associated to a  $(\frac{1}{m_{2n}^4}, 2n)$  s.c.c.  $x_n$  with  $T f_n(x_n) \geq 1/3$  and  $\text{supp } x_n \subset \text{supp } T f_n$ , and we shall also assume that  $\|T f_n\| = 1$  for each  $n$ . Build then for each  $k$  a  $(\frac{1}{m_{2k}^4}, 2k)$  R.I.s.c.c.  $y_k = \sum_{n \in A_k} a_n x_n$  such that  $(T f_n)_{n \in A_k}$

and therefore  $(f_n)_{n \in A_k}$  is  $\mathcal{M}_{2k}$ -admissible. Then note that by the first observation before this proposition,

$$c_k^* := m_{2k}^{-1} \sum_{n \in A_k} f_n \in \text{conv}_{\mathbb{Q}} L_{2k},$$

and observe that  $\text{supp } c_k^* \cap \text{supp } y_k = \emptyset$  and that  $Tc_k^*(y_k) \geq (3m_{2k})^{-1}$ .

We may therefore for any  $j > 100$  construct a  $j$ -quadruple  $(j_k, y_k, c_k^*, b_k)_{k=1}^n$  satisfying the hypotheses of Proposition 4.4 and such that  $c_k^* \in [f_i]_i$  and  $Tc_k^*(y_k) \geq (3m_{2j_k})^{-1}$  for each  $k$ . From Proposition 4.4 we deduce

$$\left\| \sum_{k=1}^n b_k m_{2j_k} y_k \right\| \leq \frac{75}{m_{2j+1}^2}.$$

Therefore

$$\left\| \sum_{k=1}^d Tc_k^* \right\| \geq \frac{\sum_{k=1}^d b_k m_{2j_k} Tc_k^*(y_k)}{\left\| \sum_{k=1}^n b_k m_{2j_k} y_k \right\|} \geq \frac{m_{2j+1}^2}{225},$$

but on the other hand

$$\left\| \sum_{k=1}^d c_k^* \right\| \leq m_{2j+1}$$

since by the second observation the functional  $m_{2j+1}^{-1} \sum_{k=1}^d c_k^*$  belongs to  $\text{conv}_{\mathbb{Q}} L$ . We deduce finally that

$$m_{2j+1} \leq 225 \|T\|,$$

which contradicts the boundedness of  $T$ .  $\square$

## 5. PROBLEMS AND COMMENTS

Obviously the general question one is compelled to ask is whether it is possible to find an example for each of the classes or subclasses appearing in the chart of Theorem 1.5. However we wish to be more specific here and concentrate on the classes which either seem particularly interesting, or easier to study, or which are related to one of the spaces considered in this paper.

Let us first observe that the examples of locally minimal, tight spaces produced so far could be said to be so for trivial reasons: since they hereditarily contain  $\ell_\infty^n$ 's uniformly, any Banach space is crudely finitely representable in any of their subspaces. It remains open whether there exist other examples. Observing that a locally minimal and tight space cannot be strongly asymptotically  $\ell_p$ ,  $1 \leq p < +\infty$ , by one of the implications in the diagram before Theorem 1.5, and up to the 6th dichotomy, the problem may be summed up as:

**Problem 5.1.** *Find a tight, locally minimal, uniformly inhomogeneous Banach space which does not contain  $\ell_\infty^n$ 's uniformly, or equivalently, which has finite co-type.*

It also unknown whether tightness by range and tightness with constants are the only possible forms of tightness, up to passing to subspaces. Equivalently, using the 4th and 5th dichotomy:

**Problem 5.2.** *Find a tight Banach space which is sequentially and locally minimal.*

Or, since such a space would have to be of type (2), (5b), (5d):

**Problem 5.3.**

- (a) Find a HI space which is sequentially minimal.
- (b) Find a space of type (5b).
- (c) Find a space of type (5d). Is the dual of some modified mixed Tsirelson's space such a space?

For the next problem, we observe that the only known examples of spaces tight with constants are strongly asymptotic  $\ell_p$  spaces not containing  $\ell_p$ , where  $1 \leq p < +\infty$ .

**Problem 5.4.** Find a space tight with constants and uniformly inhomogeneous.

More specifically, listing two subclasses for which we have a possible candidate:

**Problem 5.5.**

- (a) Find a space of type (1a). Is  $G$  or one of its subspaces such a space?
- (b) Find a space of type (3a). Is  $G_u$  or one of its subspaces such a space?

Recently, S. Argyros, K. Beanland and T. Raikoftsalis [1, 2] constructed an example  $X_{abr}$  with a basis which is strongly asymptotically  $\ell_2$  and therefore weak Hilbert, yet every operator is a strictly singular perturbation of a diagonal map, and no disjointly supported subspaces are isomorphic. In our language,  $X_{abr}$  is therefore a new space of type (3c), which we include in our chart.

We conclude by mentioning the very recent and remarkable result of S. Argyros and R. Haydon solving the scalar plus compact problem [5]: there exists a HI space which is a predual of  $\ell_1$  and on which every operator is a compact perturbation of a multiple of the identity. To our knowledge nothing is known about the exact position of this space in the chart of Theorem 1.5.

**Problem 5.6.** Find whether Argyros-Haydon's space is of type (1) or of type (2).

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